ELSEVIER Journal of Pure and Applied Algebra 112 (1996) 109–155

JOURNAL OF
PURE AND
APPLIED ALGEBRA

Automorphism groups of tree actions and of graphs of groups

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Communicated by C.A. Weibel; received 15 September 1994

Abstract

Let Γ be a group. The minimal non-abelian Γ-actions on real trees can be parametrized by the projective space of the associated length functions. The outer automorphism group of Γ, $\mathrm{Out}(\Gamma) = \mathrm{Aut}(\Gamma)/\mathrm{ad}(\Gamma)$, acts on this space. Our objective is to calculate the stabilizer $\mathrm{Out}(\Gamma)_l = \{\alpha \in \mathrm{Aut}(\Gamma) | l \circ \alpha = l\}/\mathrm{ad}(\Gamma)$, where l is the length function of a minimal non-abelian action (without inversion) on a simplicial tree. In this case, stabilizing l up to a scalar factor is equivalent to stabilizing l. The simplicial tree action is encoded by a quotient graph of groups $\mathfrak{A}$. We produce an exact sequence $1 \to \mathrm{In\,Aut}(\mathfrak{A}) \to \mathrm{Aut}(\mathfrak{A}) \to \mathrm{Out}(\Gamma)_l \to 1$. A six-step filtration on $\mathrm{Out}(\Gamma)_l$ is obtained, where successive quotients are explicitly described in terms of the data defining $\mathfrak{A}$. In the process we obtain similar information about the structure of $\mathrm{Aut}(\mathfrak{A})$. We also draw the consequences in the case of amalgams and HNN-extensions.

1991 Math. Subj. Class.: 20E08, 20F28

0. Introduction

Let Γ be a group with an action on a real tree X. The associated (hyperbolic, or translation) length function is

$$l = l_X : \Gamma \to \mathbf{R}, \qquad l(g) = \underset{x \in X}{\mathrm{Min}}\ d(gx,x). \tag{1}$$

These length functions play a role, for tree actions, like that of characters for linear representations. In particular they are class functions on Γ.

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SSDI 0022-4049(95)00134-4

It is shown in [1, 3] that, if the Γ-action on X is "minimal and non-abelian," then l_X determines X up to unique Γ-equivariant isometry (cf. Section 1.7). This permits one to parameterize such tree actions by the space of such length functions,

$$\mathrm{LF}(\Gamma)(\subset \mathbf{R}^{\mathscr{C}(\Gamma)}), \tag{2}$$

where $\mathscr{C}(\Gamma)$ denotes the set of conjugacy classes of Γ. It is natural to consider length functions only up to a scalar factor, thus forming

$$\mathrm{PLF}(\Gamma) \subset \mathbf{PR}^{\mathscr{C}(\Gamma)}. \tag{3}$$

The group $\mathrm{Aut}(\Gamma)$ acts, by pre-composition, on tree actions, and on length function. Since the latter are class functions we see that

$$\mathrm{Out}(\Gamma) = \mathrm{Aut}(\Gamma)/\mathrm{ad}(\Gamma) \text{ acts on } \mathrm{LF}(\Gamma), \tag{4}$$

and so also on $\mathrm{PLF}(\Gamma)$. The dynamics of this action has proven to be a useful tool in the study of $\mathrm{Out}(\Gamma)$.

Our object here is to calculate the stabilizer

$$\mathrm{Out}(\Gamma)_l = \{\alpha \in \mathrm{Aut}(\Gamma) \mid l \circ \alpha = l\}/\mathrm{ad}(\Gamma), \tag{5}$$

where $l = l_X$ is the length function of an action (without inversions) on a simplicial tree X, which is minimal and non-abelian. In this case, stabilizing l up to a scalar factor is equivalent to stabilizing l. Indeed, $l(\Gamma)$ has a least value $M > 0$, so if $\alpha \in \mathrm{Aut}(\Gamma)$ and $l \circ \alpha = cl$, then $cl(\Gamma) = l(\alpha\Gamma) = l(\Gamma)$, so $M = cM$, and $c = 1$.

So let X be a minimal non-abelian simplicial Γ-tree without inversions, and length function l. From the theory of simplicial tree actions (cf. [7] or [2]), the tree action (Γ, X) is encoded by a quotient graph of groups

$$\Gamma \backslash\backslash X = \mathfrak{A} = (A, \mathscr{A}). \tag{6}$$

In [2] there is introduced a notion of morphisms for graphs of groups which, in a similar fashion, encode morphisms of tree actions.

Now suppose that $\alpha \in \mathrm{Aut}(\Gamma)$ and $l \circ \alpha = l$. Then it follows from the theorem cited above that there is a unique α-equivariant isomorphism $\gamma : X \to X$. If X_α denotes X with Γ-action twisted by α, then we have an isomorphism of tree actions (α, γ) : $(\Gamma, X) \to (\Gamma, X_\alpha)$. This, by the methods of [2], can be used to produce a $\Phi \in \mathrm{Aut}(\mathfrak{A})$ which gives rise to (α, γ).

These ideas are used in Section 4 to produce an exact sequence

$$1 \to \mathrm{In\,Aut}(\mathfrak{A}) \to \mathrm{Aut}(\mathfrak{A}) \to \mathrm{Out}(\Gamma)_l \to 1. \tag{7}$$

In Section 5 we use (7) to draw some first consequences in the case of amalgams and HNN-extensions. The utility of (7) for our purposes is that, while $\mathrm{Aut}(\mathfrak{A})$ is a somewhat complicated object, it is, at the same time, very explicitly parameterized in terms of the data defining $\mathfrak{A}$, and so it is susceptible to fairly detailed computation. This is what we carry out in Sections 6 and 7. The upshot, in Theorem 8.1, is a six-step filtration on $\mathrm{Out}(\Gamma)_l$, whose successive quotients are explicitly described in terms of the data defining $\mathfrak{A}$. In the process we obtain similar information about the structure of $\mathrm{Aut}(\mathfrak{A})$.

1. Tree actions and hyperbolic length

Graphs (and trees) X here will be understood in the sense of [7] or [2]. We write VX and EX for the sets of vertices and (oriented) edges, respectively, $\partial_0 e, \partial_1 e$ for the initial and terminal vertices of $e \in EX$, and $\bar{e}$ for e with reversed orientation. For $x \in VX$ we put $E_0(x) = \{e \in EX \mid \partial_0 e = x\}$. The distance $d(x,y)$ between vertices x and y in a connected graph is the minimum length of an edge path joining them.

1.1. Γ-trees. Let Γ be a group. A *Γ-tree* is a tree X with an action of Γ on X as tree automorphisms. A *morphism* $X \to Y$ of Γ-trees is a Γ-equivariant graph morphism. We call a Γ-tree X *minimal* if it has no proper Γ-invariant subtree.

1.2. Hyperbolic length (cf. [7, 1, Section 6]). Let X be a Γ-tree and $g \in \Gamma$. Define $l_X(g)$ and $X_g \subset X$ as follows:

Inversions. If g^2 fixes a vertex but g does not then there is a unique geometric edge $\{e, \bar{e}\}$ such that $ge = \bar{e}$. We then put $l_X(g) = 0$ and $X_g = \emptyset$, and call g an *inversion*. Every $\langle g \rangle$-invariant subtree contains e. (1)

If g is not an inversion we define

$$\begin{aligned} l_X(g) &= \operatorname*{Min}_{x \in VX} d(gx, x), \\ X_g &= \{x \in VX \mid d(gx,x) = l_X(g)\}. \end{aligned} \tag{2}$$

Then X_g is the vertex set of a subtree of X, also denoted X_g. We further distinguish two cases.

Elliptic. $l_X(g) = 0$, and X_g is the tree of fixed points of g. Every $\langle g \rangle$-invariant subtree of X meets X_g. (3)

Hyperbolic. $l_X(g) > 0$. Then X_g is a linear subtree, called the *g-axis*, along which g induces a translation of amplitude $l_X(g)$. Every $\langle g \rangle$-invariant subtree contains X_g. (4)

The function $l_X : \Gamma \to \mathbf{Z}$ is called the *hyperbolic length function* of the Γ-tree X. For $g, h \in \Gamma$ we have $l_X(hgh^{-1}) = l_X(g)$ and $X_{hgh^{-1}} = hX_g$. Moreover, for $n \in \mathbf{Z}$, we have $l_X(g^n) = |n| l_X(g)$, and $X_g \subset X_{g^n}$, with equality if $n \cdot l_X(g) \neq 0$.

For $x \in VX$ and $g \in \Gamma$ put $L_x(g) = d(gx,x)$. If g is not an inversion then it follows by definition that

$$l_X(g) = \operatorname*{Min}_{x \in VX} L_x(g), \tag{5}$$

and the minimum is achieved exactly at $x \in X_g$.

1.3. Lemma. *Let $(\alpha,\gamma) : (\Gamma,X) \to (\Gamma',X')$ be a morphism of tree actions, i.e. $\alpha : \Gamma \to \Gamma'$ is a group homomorphism and $\gamma : X \to X'$ is an α-equivariant tree morphism. Let l and l' denote the corresponding hyperbolic length functions. Then, for $g \in \Gamma$, we have*

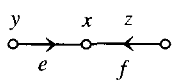
$$l'(\alpha(g)) \le l(g),$$

with equality unless g is hyperbolic on X and γ is not injective on X_g.

Proof. If g fixes $x \in VX$ then $\alpha(g)$ fixes $\gamma(x) \in VX'$. If g inverts $e \in EX$ then $\alpha(g)$ inverts $\gamma(e) \in EX'$. In both of these cases, $l(g) = 0 = l'(\alpha(g))$. Suppose finally that g on X is hyperbolic, and let $x \in VX_g$. Then

$$l(g) = d_X(gx,x) \ge d_{X'}(\gamma(gx),\gamma(x)) = d_{X'}(\alpha(g)\gamma(x),\gamma(x)) \ge l'(\alpha(g)).$$

The $\langle\alpha(g)\rangle$-invariant subtree $\gamma(X_g)$ of X' meets $X'_{\alpha(g)}$. If γ on X_g is injective, then clearly $\gamma(X_g)$ must be the $\alpha(g)$-axis, and $l'(\alpha(g)) = l(g)$. If γ on X_g is not injective, then it must fold two adjacent edges $\gamma(e) = \gamma(f)$:



Suppose that g translates X_g in the direction of e, and $l(g) = n$. If $n = 1$, then it is easy to see, by equivariance of γ, that γ folds X_g like an accordion down to a single geometric edge, which is inverted by $\alpha(g)$, whence $l(\alpha(g)) = 0$. If $n > 1$, then $z \in [gy, y]$, and so, since $\gamma(y) = \gamma(z)$,

$$\begin{aligned} l(g) &= d(gy,y) > d(gy,z) \\ &\ge d(\gamma(gy),\gamma(z)) = d(\alpha(g)\gamma(y),\gamma(y)) \\ &\ge l'(\alpha(g)). \qquad \square \end{aligned}$$

1.4. Proposition (cf. [3, Proposition 3.1]). *Let X be a Γ-tree with $l_X \neq 0$. Then there is a unique minimal Γ-invariant subtree,*

$$X_\Gamma = \bigcup_{g \in \Gamma,\, l_X(g) > 0} X_g,$$

and $l_{X_\Gamma} = l_X$.

1.5. Proposition. *If $\Gamma \leq G = \mathrm{Aut}(X)$ acts minimally on X then the centralizer,*

$$Z_G(\Gamma)\ (= \mathrm{Aut}_\Gamma(X)) = \{1\},$$

except in the following cases:

(e) *$X = \circ\!\!-\!\!\!-\!\!\!-\!\!\circ$, and $\Gamma = G$ has order 2.*

(Z) *$X \cong \mathbf{Z}$, Γ acts by translations, and $Z_G(\Gamma)$ is the full group of translations.*

Proof. Let $z \in Z_G(\Gamma)$, $z \neq 1$. If z inverts an edge e, then $\{e, \bar{e}\}$ is Γ-invariant so $X = \circ\overset{e}{-\!\!-}\circ$ (minimality), and we have case (e). If z is not an inversion then the tree X_z is Γ-invariant, so $X = X_z$ (minimality). If z is elliptic then z is the identity on $X_z = X$, contradicting $z \neq 1$. Then z is hyperbolic, so $X = X_z \cong \mathbf{Z}$. The centralizer of the translation, z, in the dihedral group $\mathrm{Aut}(X)$ is the group of translations, whence case (Z). □

1.6. Abelian actions. Let $\varphi : \Gamma \to \mathbf{Z}$ be a homomorphism. Then Γ acts on the linear tree $X(\varphi) = \mathbf{Z}$ by translation, via φ: $gn = \varphi(g) + n$ for $g \in \Gamma$, $n \in \mathbf{Z}$. Then clearly

$$l_{X(\varphi)}(g) = |\varphi(g)|.$$

Call a Γ-tree X *abelian* if $l_X = |\varphi|$ for some homomorphism $\varphi : \Gamma \to \mathbf{Z}$. It is known then that φ is unique up to a factor ± 1 ([1, (1.4)]). Moreover there is a Γ-equivariant morphism $X \to X(\varphi)$, unique up to a translation of $X(\varphi)$ [1, p.344].

For a Γ-tree X without inversions, the following conditions are equivalent (cf. [1, Section 7]):

(a) X is abelian.

(b) $l(ghg^{-1}h^{-1}) = 0$ for all $g, h \in \Gamma$ ($l = l_X$).

(c) $l(gh) \leq l(g) + l(h)$ for all $g, h \in \Gamma$.

(d) $X_g \cap X_h \neq \emptyset$ for all $g, h \in \Gamma$.

(e) Γ fixes a vertex or an end of X.

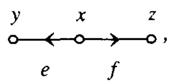
1.7. Non-abelian actions. For these we have the following uniqueness theorem.

Theorem ([1, (7.13)], or [3]). *Let X, Y be minimal non-abelian Γ-trees without inversions. If $l_X = l_Y$ then there is a unique Γ-morphism $\phi : X \to Y$, and it is an isomorphism.*

Proof. In [1, (7.13)] it is shown that if $l_X = l_Y$ then there is a (unique) Γ-isomorphism $\phi_0 : X \to Y$. It remains only to show that, if $l_X = l_Y$ and if ϕ is any Γ-morphism, then ϕ is an isomorphism, hence $\phi = \phi_0$. Since $l_X = l_Y$ we know from Lemma 1.3 that, for hyperbolic $g \in \Gamma$, $\phi : X_g \to Y_g$ is an isomorphism. Moreover it follows from [1, (7.4)] that ϕ preserves distance between hyperbolic axes. Let $g, h \in \Gamma$ be hyperbolic with disjoint axes. (These exist since X is non-abelian: [1, (7.3), (7.4) and (7.6)].) Let $[u, v] = [X_g, X_h]$ be the bridge from X_g to X_h. Then $[\phi(u), \phi(v)] = [Y_g, Y_h]$. Since both ϕ and ϕ_0 carry X_g to Y_g and X_h to Y_h, we have $\phi(u) = \phi_0(u)$. Now the locus where ϕ and ϕ_0 agree is a non-empty Γ-invariant set of vertices in X on which ϕ, like ϕ_0, is distance preserving. By minimality, this set of vertices spans X. Lemma 1.8 then shows that ϕ is an isometry on X, hence $\phi = \phi_0$. □

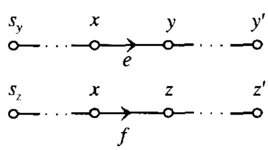
1.8. Lemma. *Let $\phi : X \to Y$ be a morphism of trees, and let $S \subset VX$ be a spanning set of vertices. (I.e. the smallest subtree of X containing S is X itself.) If $\phi|_S$ is distance preserving then ϕ on X preserves distance, and hence is injective.*

Proof. If ϕ is not injective then it "folds" two adjacent edges



, $\phi(e) = \phi(f)$.

Since S spans X, e and f belong to geometric edge paths $[s_y, y']$ and $[s_z, z']$, respectively, with $s_y, s_z, y', z' \in S$.



Then clearly $[y', z'] = [y', x] \cup [x, z']$, whereas the geodesic $[\phi(y'), \phi(z')]$, because of the fold, is contained in the shorter edge path $\phi([y', y]) \cup \phi([z, z'])$. This contradicts the fact that ϕ preserves distance on S. □

1.9. The actions of Aut(Γ) and Out(Γ). Let Γ be a group, with automorphism sequence

$$1 \to Z(\Gamma) \to \Gamma \xrightarrow{\text{ad}} \text{Aut}(\Gamma) \to \text{Out}(\Gamma) \to 1. \qquad (1)$$

Here $Z(\Gamma)$ = center of Γ, and ad(g) is the inner automorphism, sending x to gxg^{-1}.

Let X be a tree and $G = \text{Aut}(X)$. Actions of Γ on X correspond to homomorphisms $\rho \in \text{Hom}(\Gamma, G)$. Let us write here X_ρ and l_ρ for the corresponding Γ-tree and length function.

The group $\mathrm{Aut}(\Gamma)$ acts on $\mathrm{Hom}(\Gamma, G)$ by $\alpha : \rho \mapsto \rho \circ \alpha$. The stabilizer of ρ is

$$\begin{aligned}\mathrm{Aut}(\Gamma)_\rho &= \{\alpha \mid \rho \circ \alpha = \rho\}\\ &= \{\alpha \mid g^{-1}\alpha(g) \in \mathrm{Ker}(\rho) \text{ for all } g \in \Gamma\}.\end{aligned} \tag{2}$$

This is trivial when ρ is faithful (i.e. injective).

We are interested in the stabilizer of the isomorphism class (ρ) of ρ (or of X_ρ). Observe that

$$X_\rho \cong X_{\rho'} \quad \text{iff} \quad \rho' = \mathrm{ad}(\gamma) \circ \rho \quad \text{for some } \gamma \in G. \tag{3}$$

Here $\gamma : X \to X$ is the Γ-isomorphism $X_\rho \to X_{\rho'} : \gamma(\rho(g)x) = \rho'(g)\gamma(x)$ for $g \in \Gamma$, $x \in X$, i.e. $\gamma\rho(g) = \rho'(g)\gamma$ in G. Any two such γ differ by a Γ-automorphism of X_ρ. If X_ρ is minimal and non-abelian it follows from Proposition 1.5 that $\mathrm{Aut}_\Gamma(X_\rho) = \{1\}$, and so γ above is unique.

Now Theorem 1.7 in this case gives us the following result.

1.10. Theorem. *Let $\rho : \Gamma \to G = \mathrm{Aut}(X)$ define a minimal non-abelian Γ-tree X_ρ. Let $\alpha \in \mathrm{Aut}(\Gamma)$. The following conditions are equivalent.*

(a) $X_\rho \cong X_{\rho\circ\alpha}$ *(i.e. $\alpha \in \mathrm{Aut}(\Gamma)_{(\rho)}$).*

(b) $l_{\rho\circ\alpha}(= l_\rho \circ \alpha) = l_\rho$ *(i.e. $\alpha \in \mathrm{Aut}(\Gamma)_{l_\rho}$).*

(c) *There is a (unique) $\gamma \in G$ such that $\rho(\alpha(g)) = \gamma\rho(g)\gamma^{-1}$ for all $g \in \Gamma$.*

Remark. In view of (c), we have a map to the normalizer of $\rho\Gamma$, $\mathrm{Aut}(\Gamma)_{l_\rho} \to N_G(\rho\Gamma)$, $\alpha \mapsto \gamma$ which is easily seen to be a homomorphism.

1.11. Corollary. *In Theorem 1.10, suppose that ρ is the inclusion of a subgroup $\Gamma \le G$, and $l = l_\rho$. Then*

$$\mathrm{Aut}(\Gamma)_l = N_G(\Gamma),$$

the normalizer of Γ in G.

Proof. The natural homomorphism $N_G(\Gamma) \to \mathrm{Aut}(\Gamma)$ is injective, since $\mathrm{Aut}_\Gamma(X) = Z_G(\Gamma)$ is trivial, and its image is $\mathrm{Aut}(\Gamma)_{(\rho)}$, which, by Theorem 1.10, coincides with $\mathrm{Aut}(\Gamma)_l$. □

The following lemma will be used in Section 6 below.

1.12. Lemma. *Let X be a minimal non-abelian Γ-tree. Let $(\alpha, \lambda) : (\Gamma, X) \to (\Gamma, X)$ be an isomorphism of tree actions: $\alpha \in \mathrm{Aut}(\Gamma)$, $\lambda \in \mathrm{Aut}(X)$, and $\lambda(gx) = \alpha(g)\lambda(x)$*

for $g \in \Gamma$, $x \in X$. *If* $\alpha = \mathrm{ad}(u)$ *is an inner automorphism,* $u \in \Gamma$, *then* $\lambda = u$, *and hence* λ *induces the identity on* $A = \Gamma\backslash X$.

Proof. Since $u : X \to X$ is also equivariant for $\alpha = \mathrm{ad}(u)$, we have $\lambda = uv$ with $v \in \mathrm{Aut}_\Gamma(X)$. When X is minimal non-abelian we have $\mathrm{Aut}_\Gamma(X) = \{1\}$, by Proposition 1.5, whence $\lambda = u$. □

2. Graphs of groups and length functions

2.1. A graph of groups $\mathfrak{A} = (A, \mathscr{A})$ consists of a connected graph A, groups $\mathscr{A}_a$ $(a \in VA)$, and $\mathscr{A}_e = \mathscr{A}_{\bar{e}}$ $(e \in EA)$, and monomorphisms $\alpha_e : \mathscr{A}_e \to \mathscr{A}_{\partial_0 e}$. The *path group* is

$$\pi(\mathfrak{A}) = [(\mathop{*}_{a \in VA} \mathscr{A}_a) * F(EA)]/N$$

where $F(EA)$ is the free group with basis EA, and N is the normal subgroup that imposes the relations

$$e\bar{e} = 1$$

and

$$e\alpha_{\bar{e}}(s)e^{-1} = \alpha_e(s)$$

for all $e \in EA$, $s \in \mathscr{A}_e$. We identify $\mathscr{A}_a$ and EA with their images in $\pi(\mathfrak{A})$ (cf. [2, Section 1]).

2.2. Paths in $\mathfrak{A}$. A path in $\mathfrak{A}$ is a finite sequence

$$\gamma = (g_0, e_1, g_1, e_2, \ldots, g_{n-1}, e_n, g_n), \tag{1}$$

where $(e_1, \ldots, e_n)$ is an edge-path in A, say $\partial_1 e_i = a_i = \partial_0 e_{i+1}$ $(1 \le i < n)$, $a_0 = \partial_0 e_1$, $a_n = \partial_1 e_n$, and we have $g_i \in \mathscr{A}_{a_i}$ $(0 \le i \le n)$. We call γ a *path of length* n *from* a_0 *to* a_n, and put

$$|\gamma| = g_0 e_1 g_1 e_2 \cdots g_{n-1} e_n g_n \in \pi(\mathfrak{A}). \tag{2}$$

For $a, b \in VA$ let

$$P[a,b] = \text{the set of paths (in } \mathfrak{A}\text{) from } a \text{ to } b, \tag{3}$$

and

$$\pi[a,b] = |P[a,b]| \subset \pi(\mathfrak{A}). \tag{4}$$

For $g \in \pi[a,b]$ define the length

$$L_{\mathfrak{A}}(g) = \min\{\mathrm{length}(\gamma) \mid \gamma \in P[a,b], |\gamma| = g\}. \tag{5}$$

Note that $L_{\mathfrak{A}} : \pi[a,b] \to \mathbf{Z}$ factors through $\mathscr{A}_a \backslash \pi[a,b] / \mathscr{A}_b$.

With $\gamma \in P[a,b]$ as above $(a = a_0,\ b = a_n)$ and

$$\delta = (h_0, f_1, h_1, \ldots, h_{m-1}, f_m, h_m) \in P[b,c]$$

we can define the composite $\gamma\delta \in P[a,c]$ by

$$\gamma\delta = (g_0, e_1, g_1, \ldots, e_n, g_n h_0, f_1, h_1, \ldots, f_m, h_m).$$

Clearly $|\gamma\delta| = |\gamma||\delta|$. Whence a product

$$\pi[a,b] \times \pi[b,c] \to \pi[a,c] \tag{6}$$

given by multiplication in $\pi(\mathfrak{A})$.

Defining

$$\gamma^{-1} = (g_n^{-1}, \bar{e}_n, g_{n-1}^{-1}, \ldots, g_1^{-1}, \bar{e}_1, g_0^{-1}) \in P[b,a], \tag{7}$$

we have $|\gamma^{-1}| = |\gamma|^{-1}$, whence

$$\pi[b,a] = \pi[a,b]^{-1}. \tag{8}$$

Thus we have the *fundamental group at a*,

$$\Gamma_a = \pi_1(\mathfrak{A}, a) := \pi[a,a]. \tag{9}$$

It is easily seen that, for $g \in \pi[a,b]$, we have

$$\Gamma_a \cdot g = \pi[a,b] = g \cdot \Gamma_b. \tag{10}$$

Let $T \subset A$ be a spanning tree, and put

$$\pi_1(\mathfrak{A}, T) = \pi(\mathfrak{A})/(e = 1 \text{ for all } e \in ET). \tag{11}$$

Then (cf. [2, (1.20)]) the projection

$$q : \pi(\mathfrak{A}) \to \pi_1(\mathfrak{A}, T)$$

restricts, for each $a \in VA$, to an *isomorphism*

$$q_a : \pi_1(\mathfrak{A}, a) \xrightarrow{\cong} \pi_1(\mathfrak{A}, T). \tag{12}$$

The inverse σ_a of q_a is given as follows. For $a, b \in VA$, let $\gamma_{a,b} = (e_1, \ldots, e_n)$ denote the edge path in T from a to b, and put $g_{a,b} = |\gamma_{a,b}| = e_1 \cdots e_n \in \pi[a,b]$. Then σ_a is given on generators by $\sigma_a(s) = g_{a,b} s g_{a,b}^{-1}$ for $s \in \mathscr{A}_b$, and $\sigma_a(e) = g_{a,\partial_0 e} e g_{a,\partial_1 e}^{-1}$ for $e \in EA$. Since $g_{a,b} g_{b,c} = g_{a,c}$, it follows that the following diagram is commutative:

$$\begin{array}{ccc} \pi_1(\mathfrak{A},a) & \xrightarrow{\operatorname{ad}(g_{b,a})} & \pi_1(\mathfrak{A},b) \\ \sigma_a \uparrow & & \uparrow \sigma_b \\ \pi_1(\mathfrak{A},T) & = & \pi_1(\mathfrak{A},T) \end{array} \tag{13}$$

2.3. The covering tree $X_a = (\widetilde{\mathfrak{A}, a})$, at a base point $a \in VA$, has vertices

$$VX_a = \coprod_{b \in VA} \pi[a,b]/\mathscr{A}_b.$$

For $g \in \pi[a,b]$ we let $[g]_b$ denote its class in $\pi[a,b]/\mathscr{A}_b$. The group $\Gamma_a = \pi_1(\mathfrak{A}, a)$ acts on X_a so that $g[h]_b = [gh]_b$ for $g \in \Gamma_a$, $h \in \pi[a,b]$. The orbits are the sets $\pi[a,b]/\mathscr{A}_b$, which are also the fibers of the projection $p : X_a \to A = \Gamma_a \backslash X_a$.

To calculate the length function

$$l_a (= l_{X_a}) : \Gamma_a = \pi_1(\mathfrak{A}, a) \to \mathbf{Z}$$

we use the following result from ([4, Lemma 1.1]):

For $x = [g]_b \in \pi[a,b]/\mathscr{A}_b$ and $y = [h]_c \in \pi[a,c]/\mathscr{A}_c$, their distance in X_a is given by

$$d_{X_a}(x,y) = L_{\mathfrak{A}}(g^{-1}h), \tag{1}$$

where $g^{-1}h \in \pi[b,c]$. For $a = b = c$ and $g = 1$, so that $x = [1]_a$, this gives $L_{\mathfrak{A}}(h) = d(hx,x)$ for $h \in \Gamma_a$.

Now for $g \in \Gamma_a$ and $x = [h]_b \in \pi[a,b]/\mathscr{A}_b$, we have $d(x,gx) = d([h]_b,[gh]_b) = L_{\mathfrak{A}}(h^{-1}gh)$. Now from 1.2(5) it follows that

$$l_a(g) = \operatorname*{Min}_{\substack{b \in VA \\ h \in \pi[a,b]}} L_{\mathfrak{A}}(h^{-1}gh). \tag{2}$$

If $g \in \pi[b,a]$ then we have an isomorphism of tree actions

$$(\mathrm{ad}(g), g\cdot) : (\Gamma_a, X_a) \to (\Gamma_b, X_b) \tag{3}$$

given by $\mathrm{ad}(g)(h) = ghg^{-1}$ for $h \in \Gamma_a$, and $g \cdot [h]_c = [gh]_c$ for $h \in \pi[a,c]$. (Cf. [2, (1.22)]). It follows then from Section 1.3 that, for $h \in \Gamma_a$,

$$l_b(ghg^{-1}) = l_a(h). \tag{4}$$

2.4. Quotient graphs of groups (cf. [2, Section 3]). Let X be a Γ-tree without inversions. The construction of a "quotient graph of groups"

$$\Gamma \backslash\backslash X = \mathfrak{A} = (A, \mathscr{A})$$

depends on choosing subtrees

$$T \subset S \subset X,$$

and elements $(g_x)_{x \in VS}$ of Γ, so that, if $p : X \to A := \Gamma \backslash X$ is the natural projection, then $p : VT \to VA$ is bijective, $p : ES \to EA$ is bijective, and $g_x x \in VT$ for all $x \in VS$, with $g_x = 1$ if $x \in VT$. Denoting the inverses of the above bijections by $a \mapsto a^X$ and $e \mapsto e^X$, respectively, we have $\mathscr{A}_a = \Gamma_{a^X}$, $\mathscr{A}_e = \Gamma_{e^X}$, and, if $\partial_0(e) = a$ and $\partial_0(e^X) = x$, then $\alpha_e = \mathrm{ad}(g_x) : \mathscr{A}_e \to \mathscr{A}_a$.

The homomorphism $\psi : \pi(\mathfrak{A}) \to \Gamma$ is then defined on generators by $\psi(g) = g$ for $g \in \mathscr{A}_a = \Gamma_{a^X}$, and $\psi(e) = g_e := g_0 g_1^{-1}$ for $e \in EA$, where $g_i = g_{\partial_i(e^X)}$ $(i = 0, 1)$. Then ψ restricts to isomorphisms $\psi_a : \Gamma_a = \pi_1(\mathfrak{A}, a) \to \Gamma$ for each $a \in VA$.

There is further a ψ_a-equivariant isomorphism of trees, $\tau_\alpha : X_a = (\widetilde{\mathfrak{A}, a}) \to X$ defined on $[g]_b \in \pi[a,b]_b \subset VX_a$ by $\tau_a([g]_b) = \psi(g) \cdot b^X$. Thus we have an isomorphism of tree actions,

$$(\psi_a, \tau_a) : (\Gamma_a, X_a) \to (\Gamma, X).$$

2.5. Adapting to an automorphism. Keep the notation of 2.4 above, and let $\rho : \Gamma \to G = \mathrm{Aut}(X)$ define the given Γ-action on X. Let $\alpha \in \mathrm{Aut}(\Gamma)$, and let X_α denote the tree X with Γ-action defined by $\rho \circ \alpha$.

Suppose that $\alpha \in \mathrm{Aut}(\Gamma)_{(\rho)}$. This means that there is a $\lambda \in G$ which is a Γ-isomorphism $\lambda : X \to X_\alpha : \lambda(\rho(g)x) = \rho(\alpha(g))\lambda(x)$, for $g \in \Gamma$ and $x \in X$. Thus we have the stabilizers

$$\Gamma_{\rho,x} = \Gamma_{\rho\circ\alpha,\lambda(x)} \tag{1}$$

where $\Gamma_{\rho,x} = \{g \in \Gamma \mid \rho(g)x = x\}$, and similarly for $\Gamma_{\rho\circ\alpha,\lambda(x)}$.

Let $T \subset S \subset X$ and $(g_x)_{x \in VS}$ be the fundamental data as in 2.4 above used to construct

$$\Gamma \backslash\backslash X = \mathfrak{A} = (A, \mathscr{A}).$$

Then we can use $\lambda T \subset \lambda S \subset X_\alpha$ as fundamental domains for the $\rho \circ \alpha$-action. Further, for $x \in VS$, we have $g_x \cdot x \in T$ (and $g_x = 1$ for $x \in VT$), so $\rho(\alpha(g_x))\lambda(x) = \lambda(\rho(g_x)x) \in V\lambda T$ (and $g_x = 1$ for $\lambda x \in V\lambda T$). Thus, defining $g'_{\lambda x} = g_x$, we can use $(g'_{\lambda x})_{\lambda x \in V\lambda S}$ in defining $\mathfrak{A}' = \Gamma \backslash\backslash X_\alpha$. It then follows from the construction (see 2.4) that

$$\mathfrak{A}' = \mathfrak{A}!$$

In fact, for $a \in VA$ and $e \in EA$ let $(a^X)'$ and $(e^X)'$ denote their lifts to $V\lambda T$ and $E\lambda S$, respectively. Then $(a^X)' = \lambda a^X$ and $(e^X)' = \lambda e^X$, clearly. Further,

$$g'_i := g'_{\partial_i(e^X)'} = g'_{\partial_i \lambda e^X} = g'_{\lambda \partial_i e^X} = g_{\partial_i e^X} = g_i.$$

Hence, if $a = \partial_0 e$, then

$$\alpha_e = \mathrm{ad}(g_0) : \mathscr{A}_e = \Gamma_{\rho,e^X} \to \mathscr{A}_a = \Gamma_{\rho,a^X}$$

coincides with

$$\alpha'_e = \mathrm{ad}(g'_0) : \mathscr{A}'_e = \Gamma_{\rho\circ\alpha,\lambda e^X} \to \mathscr{A}'_a = \Gamma_{\rho\circ\alpha,\lambda a^X}.$$

Further, the homomorphisms $\psi : \pi(\mathfrak{A}) \to \Gamma$ and $\psi' : \pi(\mathfrak{A}') \to \Gamma$ are both the inclusion on $\mathscr{A}_a = \mathscr{A}'_a$, and on $e \in EA$ as above,

$$\psi'(e) = g'_0 {g'}_1^{-1} = g_0 g_1^{-1} = \psi(e).$$

Thus

$$\psi' = \psi : \pi(\mathfrak{A}) \to \Gamma.$$

For $a \in VA$ put $\Gamma_a = \pi_1(\mathfrak{A}, a) = \Gamma'_a$ and $X_a = (\widetilde{\mathfrak{A}, a}) = X'_a$. Then we have tree isomorphisms

$$\tau_a : X_a \to X \quad \text{and} \quad \tau'_a : X'_a \to X_\alpha$$

which are equivariant for $\psi_a : \Gamma_a \to \Gamma$. Let $[g]_b \in \pi[a,b]/\mathscr{A}_b \subset VX_a$. Then, by definition (cf. 2.4),

$$\tau_a([g]_b) = \psi(g) \cdot b^X$$

and

$$\begin{aligned} \tau'_a([g]_b) &= \rho(\alpha(\psi(g)))(b^X)' \\ &= \rho(\alpha(\psi(g)))\lambda(b^X) \\ &= \lambda(\rho(\psi(g))b^X) \\ &= \lambda(\tau_a([g]_b)). \end{aligned}$$

Thus we have a commutative diagram

$$\begin{array}{ccc} X_a & = & X'_a \\ {\scriptstyle \tau_a}\downarrow & & \downarrow{\scriptstyle \tau'_a} \\ X & \xrightarrow[\lambda]{} & X_a \end{array} .$$

2.6 Reduced paths. Let

$$\gamma = (g_0, e_1, g_1, \ldots, g_{n-1}, e_n, g_n) \tag{1}$$

be a path in $\mathfrak{A}$, with vertex sequence $a_0, a_1, \ldots, a_n$, as in 2.2(1). We call γ *reduced* if, for $i = 1, \ldots, n-1$, either $e_{i+1} \neq \bar{e}_i$ or $e_{i+1} = \bar{e}_i$ and $g_i \notin \alpha_{\bar{e}_i}(\mathscr{A}_{e_i})$. When $a_0 = a_n$ we call γ *cyclically reduced* if it is reduced, and either $e_n \neq \bar{e}_1$, or $e_n = \bar{e}_1$ and $g_n g_0 \notin \alpha_{e_1}(\mathscr{A}_{e_1})$.

If $g \in \pi[a,b]$ then $g = |\gamma|$ for a reduced path $\gamma \in P[a,b]$, and length$(\gamma) = L_{\mathfrak{A}}(g)$ for any such γ (cf. [2, (1.10)]).

2.7. Lemma. *For any closed path γ in $\mathfrak{A}$, there are paths γ_1, γ_2 such that $|\gamma| = |\gamma_1 \gamma_2 \gamma_1^{-1}|$, γ_1 is reduced, and γ_2 is cyclically reduced.*

Proof. Let

$$\gamma = (g_0, e_1, g_1, e_2, \ldots, g_{n-1}, e_n, g_n).$$

If γ is cyclically reduced, then let $\gamma_2 = \gamma$, and let $|\gamma_1| = 1$. Now suppose that γ is reduced, but not cyclically reduced. We prove the lemma by induction on $L_{\mathfrak{A}}(|\gamma|) = n$. Since γ is not cyclically reduced, $e_1 = \overline{e}_n$ and

$$g_n g_0 = \alpha_{\overline{e}_n}(s) \in \alpha_{\overline{e}_n}(\mathscr{A}_{e_n}).$$

Since $e\alpha_{\overline{e}}(s)\overline{e} = \alpha_e(s)$ for all $s \in \mathscr{A}_e$,

$$e_n g_n g_0 e_1 = e_n g_n g_0 \overline{e}_n = e_n \alpha_{\overline{e}_n}(s)\overline{e}_n = \alpha_{e_n}(s).$$

Let

$$\begin{aligned}\gamma' &= (g_1, e_2, \dots, e_{n-1}, g_{n-1} e_n g_n g_0 e_1)\\ &= (g_1, e_2, \dots, e_{n-1}, g_{n-1}\alpha_{e_n}(s)).\end{aligned}$$

Then $L_{\mathfrak{A}}(|\gamma'|) = n-2$. By induction, $|\gamma'| = |\gamma_1'\gamma_2'\gamma_1'^{-1}|$ for some paths γ_1' and γ_2', where γ_2' is cyclically reduced. So

$$\begin{aligned}|\gamma| &= (g_0 e_1)(g_1 e_2 \cdots e_{n-1} g_{n-1} e_n g_n g_0 e_1)(g_0 e_1)^{-1}\\ &= (g_0 e_1)(|\gamma'|)(g_0 e_1)^{-1}\\ &= (g_0 e_1)|\gamma_1'\gamma_2'\gamma_1'^{-1}|(g_0 e_1)^{-1}.\end{aligned}$$

Let γ_1 be a reduced path representing $(g_0 e_1)|\gamma_1'|$, and let $\gamma_2 = \gamma_2'$. Then

$$|\gamma| = |\gamma_1\gamma_2\gamma_1^{-1}|,$$

where γ_2 is cyclically reduced. □

3. The category of graphs of groups

This section is a resume of material from [2, Section 2].

3.1. Morphisms of graphs of groups

3.1. Morphisms of graphs of groups (cf. [2, Section 2]). A morphism

$$\Phi = (\phi, (\gamma)) : \mathfrak{A} = (A, \mathscr{A}) \to \mathfrak{A}' = (A', \mathscr{A}')$$

of graphs of groups consists of a graph morphism ϕ (or ϕ_A) : $A \to A'$, group homomorphisms

$$\phi_a : \mathscr{A}_a \to \mathscr{A}'_{\phi(a)} \quad (a \in VA) \quad \text{and} \quad \phi_e = \phi_{\overline{e}} : \mathscr{A}_e \to \mathscr{A}'_{\phi(e)} \quad (e \in EA),$$

and families $(\gamma_a)_{a\in VA}$, $(\gamma_e)_{e\in EA}$ in $\pi(\mathfrak{A}')$, satisfying the following conditions.

For $a \in VA$, $\gamma_a \in \pi_1(\mathfrak{A}', \phi(a))$. For $e \in EA$, $\partial_0 e = a$, we have $\delta_e := \gamma_a^{-1}\gamma_e \in \mathscr{A}'_{\phi(a)}$, and the following diagram commutes:

$$\begin{array}{ccc} \mathscr{A}_a & \xrightarrow{\operatorname{ad}(\delta_e^{-1})\circ\phi_a} & \mathscr{A}'_{\phi(a)} \\ \uparrow{\scriptstyle \alpha_e} & & \uparrow{\scriptstyle \alpha'_{\phi(e)}} \\ \mathscr{A}_e & \xrightarrow[\phi_e]{} & A'_{\phi(e)} \end{array} \tag{1}$$

The *identity morphism* of $\mathfrak{A}$ is $I = (\phi, (\gamma))$ given by $\phi_A = \mathrm{Id}_A$, $\phi_u = \mathrm{Id}_{\mathscr{A}_u}$, and $\gamma_u = 1$ for $u \in VA \cup EA$.

3.2. The induced homomorphism

$$\Phi \text{ (or } \Phi_\pi) : \pi(\mathfrak{A}) \to \pi(\mathfrak{A}')$$

is defined on generators by $\Phi|_{\mathscr{A}_a} = \operatorname{ad}(\gamma_a) \circ \phi_a$, i.e. $\Phi(s) = \gamma_a \phi_a(s) \gamma_a^{-1}$ for $s \in \mathscr{A}_a$, and $\Phi(e) = \gamma_e \phi(e) \gamma_{\bar{e}}^{-1}$ for $e \in EA$. For $a, b \in VA$, Φ restricts to maps

$$\Phi : \pi^{\mathfrak{A}}[a,b] \to \pi^{\mathfrak{A}'}[\phi(a), \phi(b)].$$

In particular, for $a = b$, we have the homomorphism

$$\Phi_a : \pi_1(\mathfrak{A}, a) \to \pi_1(\mathfrak{A}', \phi(a)).$$

3.3. The tree morphism

$$\tilde{\Phi}_a : (\widetilde{\mathfrak{A}, a}) \to (\widetilde{\mathfrak{A}', \phi(a)}),$$

which is Φ_a-equivariant, is defined on the vertices $\pi^{\mathfrak{A}}[a,b]/\mathscr{A}_b$ by

$$\tilde{\Phi}_a([g]_b) = [\Phi(g)\gamma_a]_{\phi(a)}.$$

Thus we have a morphism of tree actions

$$(\Phi_a, \tilde{\Phi}_a) : (\Gamma_a, X_a) \to (\Gamma'_{\phi(a)}, X'_{\phi(a)}),$$

where $\Gamma_a = \pi_1(\mathfrak{A}, a)$, $X_a = (\widetilde{\mathfrak{A}, a})$, and similarly for $\Gamma'_{\phi(a)}$ and $X'_{\phi(a)}$.

3.4. $\delta\Phi = (\phi, (\delta))$, **and the path map.** The morphism $\delta\Phi$ is obtained by preserving ϕ and δ_e $(e \in EA)$, but "suppressing" all γ_a $(a \in VA)$. Thus $\delta\Phi = (\phi, (\gamma'))$, where $\gamma'_a = 1$ $(a \in VA)$, and $\gamma'_e = \delta_e = \delta'_e$ $(e \in EA)$ (cf. [2, (2.9)]). We have [2, (2.9)]

$$\Phi_a = \operatorname{ad}(\gamma_a) \circ (\delta\Phi)_a : \pi_1(\mathfrak{A}, a) \to \pi_1(\mathfrak{A}', \phi(a)). \tag{1}$$

Evidently,

$$\delta(\delta\Phi) = \delta\Phi. \tag{2}$$

For a path $\gamma = (g_0, e_1, \ldots, e_n, g_n)$ in $\mathfrak{A}$ we define the path

$$\delta\Phi(\gamma) = (\phi_{a_0}(g_0)\delta_{e_1}, \phi(e_1), \delta_{\bar{e}_1}^{-1}\phi_{a_1}(g_1)\delta_{e_2}, \phi(e_2), \ldots, \phi(e_n), \delta_{\bar{e}_n}^{-1}\phi_{a_n}(g_n)). \tag{3}$$

Note that

$$\delta_{\bar{e}_i}^{-1} \in \mathscr{A}'_{\phi(\partial_0\bar{e}_i)} = \mathscr{A}'_{\phi(\partial_1 e_i)} = \mathscr{A}'_{\partial_1\phi(e_i)},$$

$$\phi_{a_i}(g_i) \in \mathscr{A}'_{\phi(\partial_1 e_i)} = \mathscr{A}'_{\partial_1\phi(e_i)},$$

$$\delta_{e_{i+1}} \in \mathscr{A}'_{\phi(\partial_0 e_{i+1})} = \mathscr{A}'_{\partial_1\phi(e_i)}.$$

So

$$\delta_{\bar{e}_i}^{-1}\phi_{a_i}(g_i)\delta_{e_{i+1}} \in \mathscr{A}'_{\partial_1\phi(e_i)} \quad (1 \le i \le n-1),$$

$\phi_{a_0}(g_0)\delta_{e_1} \in \mathscr{A}'_{\partial_0\phi(e_1)}$, and $\delta_{\bar{e}_n}^{-1}\phi_{a_n}(g_n) \in \mathscr{A}'_{\partial_1\phi(e_n)}$. Note that $(\phi(e_1), \ldots, \phi(e_n))$ is an edge path in A'. Thus

$$\delta\Phi(\gamma) \text{ is a path in } \mathfrak{A}', \text{ and } |\delta\Phi(\gamma)| = (\delta\Phi)(|\gamma|). \tag{4}$$

Further,

$$\text{If } g \in \pi^{\mathfrak{A}}[a,b] \text{ then } (\delta\Phi)(g) \in \pi^{\mathfrak{A}'}[\phi(a), \phi(b)] \text{ and } L_{\mathfrak{A}'}((\delta\Phi(g)) \le L_{\mathfrak{A}}(g). \tag{5}$$

In fact, we can write $g = |\gamma|$ with γ reduced. Then $L_{\mathfrak{A}}(g) = \text{length}(\gamma)$, while

$$L_{\mathfrak{A}'}(\delta\Phi(g)) \le \text{length}(\delta\Phi(\gamma)) = \text{length}(\gamma).$$

3.5. Lemma. *Assume that*

(i) ϕ_A *is injective,*

(ii) ϕ_a *is injective* $\forall a \in VA$, *and*

(iii) ϕ_e *is an isomorphism* $\forall e \in EA$.

Then $\delta\Phi$ *preserves* (*cyclically*) *reduced paths.*

Proof. Let $\gamma = (e_1, g_1, e_2)$ be a reduced path, and let $\partial_0 e_2 = a$. We only need to show that

$$\delta\Phi(\gamma) = (\phi(e_1), \delta_{\bar{e}_1}^{-1}\phi_a(g_1)\delta_{e_2}, \phi(e_2))$$

is still a reduced path. Since γ is reduced, either $e_1 \ne \bar{e}_2$, or else $e_1 = \bar{e}_2$ and $g_1 \notin \alpha_{\bar{e}_1}(\mathscr{A}_{e_1})$. Recall that ϕ is injective. If $e_1 \ne \bar{e}_2$, then $\phi(e_1) \ne \phi(\bar{e}_2)$, and $\delta\Phi(\gamma)$ is reduced. Now suppose that $e_1 = \bar{e}_2$ and $g_1 \notin \alpha_{\bar{e}_1}(\mathscr{A}_{e_1})$. Since $e_1 = \bar{e}_2$, $\phi(e_1) = \phi(\bar{e}_2)$. If $\delta\Phi(\gamma)$ is not reduced, then

$$\delta_{\bar{e}_1}^{-1}\phi_a(g_1)\delta_{e_2} = \alpha_{\phi(\bar{e}_1)}(s)$$

for some $s \in \mathscr{A}'_{\phi(e_1)}$. Thus

$$\phi_a(g_1) = \delta_{\bar{e}_1}\alpha_{\phi(\bar{e}_1)}(s)\delta_{e_2}^{-1} = \delta_{e_2}\alpha_{\phi(e_2)}(s)\delta_{e_2}^{-1}.$$

Since ϕ_{e_2} is an isomorphism,

$$s \in \mathscr{A}'_{\phi(e_1)} = \mathscr{A}'_{\phi(e_2)} = \phi_{e_2}(\mathscr{A}_{e_2}).$$

Suppose that $s = \phi_{e_2}(s_1)$ for some $s_1 \in \mathscr{A}_{e_2}$. Then

$$\phi_a(g_1) = \delta_{e_2}\alpha_{\phi(e_2)}(s)\delta_{e_2}^{-1} = \delta_{e_2}\alpha_{\phi(e_2)}(\phi_{e_2}(s_1))\delta_{e_2}^{-1}.$$

By 3.1(1),

$$\delta_{e_2}\alpha_{\phi(e_2)}(\phi_{e_2}(s_1))\delta_{e_2}^{-1} = \phi_a(\alpha_{e_2}(s_1)).$$

So $\phi_a(g_1) = \phi_a(\alpha_{e_2}(s_1))$. Since ϕ_a is injective,

$$g_1 = \alpha_{e_2}(s_1) \in \alpha_{e_2}(\mathscr{A}_{e_2}) = \alpha_{\bar{e}_1}(\mathscr{A}_{e_1}),$$

which contradicts the fact that γ is reduced. Thus $\delta\Phi(\gamma)$ is reduced. □

3.6. The composition of morphisms (cf. [2, (2.11)]),

$$\mathfrak{A} \xrightarrow{\Phi=(\phi,(\gamma))} \mathfrak{A}' \xrightarrow{\Phi'=(\phi',(\gamma'))} \mathfrak{A}'', \tag{1}$$

is given by

$$\Phi'' = \Phi' \circ \Phi = (\phi'', (\gamma'')) : \mathfrak{A} \to \mathfrak{A}'', \tag{2}$$

defined by $\phi''_A = \phi'_{A'} \circ \phi_A$, and, for $u \in VA \cup EA$, $\phi''_u = \phi'_{\phi(u)} \circ \phi_u$, and $\gamma''_u = \Phi'(\gamma_u)\gamma'_{\phi(u)}$. From [2, (2.11)], we have, for $e \in E_0(a)$,

$$\delta''_e = \phi'_{\phi(a)}(\delta_e) \cdot \delta'_{\phi(e)}. \tag{3}$$

We further have

$$\Phi''_\pi = \Phi'_\pi \circ \Phi_\pi : \pi(\mathfrak{A}) \to \pi(\mathfrak{A}'') \tag{4}$$

and, for $a \in VA$,

$$(\Phi''_a, \tilde{\Phi}''_a) = (\Phi'_{\phi(a)}, \tilde{\Phi}'_{\phi(a)}) \circ (\Phi_a, \tilde{\Phi}_a) : (\Gamma_a, X_a) \to (\Gamma''_{\phi''(a)}, X''_{\phi''(a)}) \tag{5}$$

where we write $\Gamma_a = \pi_1(\mathfrak{A}, a)$, $X_a = (\widetilde{\mathfrak{A}, a})$, and similarly for $\Gamma''_{\phi''(a)}$ and $X''_{\phi''(a)}$.

We further note that

$$\delta(\Phi' \circ \Phi) = \delta\Phi' \circ \delta\Phi. \tag{6}$$

To see this, put $\delta\Phi = (\phi, (\delta))$, $\delta\Phi' = (\phi', (\delta'))$, and $\delta\Phi'' = (\phi'', (\delta''))$. The composition formulas for ϕ''_A and ϕ''_u ($u \in VA \cup EA$) are unaffected by δ, hence still valid. Thus the only thing to be checked is that, for $e \in E_0(a)$, $a \in VA$, we have

$$\delta''_e = (\delta\Phi')_{\phi_A(a)}(\delta_e) \cdot \delta'_{\phi_A(e)}. \tag{7}$$

Since $\delta_e \in \mathscr{A}'_{\phi_A(a)}$ and $(\delta\Phi')_{\phi_A(a)}|_{\mathscr{A}'_{\phi(a)}} = \phi'_{\phi_A(a)}$, (7) follows from (3).

The above notions of morphism and composition make graphs of groups the objects of a category, with identity morphisms as in 3.1. In particular,

$$\Phi \text{ is an isomorphism iff } \phi_A \text{ and each } \phi_u \quad (u \in VA \cup EA) \text{ is an isomorphism.} \tag{8}$$

In this case,

$$\begin{aligned} &\Phi^{-1} = (\phi', (\gamma')) \text{ is given by } \phi'_A = \phi_A^{-1}, \\ &\text{and, for } u \in VA \cup EA,\ \phi'_u = \phi_u^{-1}, \text{ and } \gamma'_{\phi(u)} = \Phi^{-1}(\gamma_u)^{-1}. \end{aligned} \tag{9}$$

3.7. The group Aut($\mathfrak{A}$) is now defined, and we have the exact sequence

$$1 \to \mathrm{Aut}^A(\mathfrak{A}) \to \mathrm{Aut}(\mathfrak{A}) \xrightarrow{q_A} \mathrm{Aut}(A),$$

where, for $\Phi = (\phi, (\gamma))$, $q_A(\Phi) = \phi_A$. Thus $\Phi \in \mathrm{Aut}^A(\mathfrak{A})$ iff $\phi_A = \mathrm{Id}_A$, in which case $\phi_u \in \mathrm{Aut}(\mathscr{A}_u)$ for all $u \in VA \cup EA$. We have further a homomorphism

$$\mathrm{Aut}^A(\mathfrak{A}) \xrightarrow{q} \left[\prod_{a \in VA} \mathrm{Aut}(\mathscr{A}_a) \times \prod_{e \in EA} \mathrm{Aut}(\mathscr{A}_e) \right]$$

given, on $\Phi = (\phi, (\gamma))$, by

$$q(\Phi) = ((\phi_a)_{a \in VA}, (\phi_e)_{e \in EA}).$$

3.8. The homomorphism $\sigma_a : \mathrm{Aut}(\mathfrak{A}) \to \mathrm{Out}(\Gamma_a)_{l_a}$. For $a \in VA$ we have $\Gamma_a = \pi_1(\mathfrak{A}, a)$, the Γ_a-tree $X_a = (\widetilde{\mathfrak{A}, a})$, and its hyperbolic length function $l_a = l_{X_a}$.

Fix a spanning tree $T \subset A$. For $a, b \in VA$ let $\gamma_{a,b} = (e_1, \ldots, e_n)$ denote the reduced edge-path in T from a to b, and put $g_{a,b} = |\gamma_{a,b}| = e_1 \cdots e_n \in \pi[a,b]$. Note that $g_{a,b} g_{b,c} = g_{a,c}$. Further, from 2.3(3) we have an isomorphism of tree actions,

$$(\mathrm{ad}(g_{b,a}), g_{b,a} \cdot) : (\Gamma_a, X_a) \to (\Gamma_b, X_b). \tag{1}$$

Let $\Phi = (\phi, (\gamma)) \in \mathrm{Aut}(\mathfrak{A})$. Then from (1) and 3.3 we have the isomorphisms of group actions

$$(\Gamma_a, X_a) \xrightarrow{(\Phi_a, \tilde{\Phi}_a)} (\Gamma_{\phi(a)}, X_{\phi(a)}) \xrightarrow{(\mathrm{ad}(g), g \cdot)} (\Gamma_a, X_a) \tag{2}$$

where $g = g_{a,\phi(a)}$. This yields

$$\begin{aligned} \Phi_{(a)} &:= \mathrm{ad}(g_{a,\phi(a)}) \circ \Phi_a, \\ \tilde{\Phi}_{(a)} &:= (g_{a,\phi(a)} \cdot) \circ \tilde{\Phi}_a \end{aligned} \tag{3}$$

so that

$$(\Phi_{(a)}, \tilde{\Phi}_{(a)}) : (\Gamma_a, X_a) \to (\Gamma_a, X_a) \text{ is an isomorphism of tree actions.} \tag{4}$$

It follows from Lemma 1.3 that

$$\Phi_{(a)} \text{ preserves the length function } l_a \tag{5}$$

From the commutative diagram

$$\begin{array}{ccccc}
X_a & \xrightarrow{\bar{\Phi}_a} & X_{\phi(a)} & \xrightarrow{g_{a,\phi(a)}} & X_a \\
\downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} \\
A & \xrightarrow[\phi_A]{} & A & \xrightarrow[\mathrm{Id}_A]{} & A
\end{array}$$

we see that

$$\tilde{\Phi}_{(a)} \text{ induces } \phi_A \text{ on } A = \Gamma_a \backslash X_a. \tag{6}$$

Let $b \in VA$. We have a commutative diagram

$$\begin{array}{ccccc}
\Gamma_a & \xrightarrow{\Phi_a} & \Gamma_{\phi(a)} & \xrightarrow{\mathrm{ad}(g_{a,\phi(a)})} & \Gamma_a \\
\downarrow{\scriptstyle \mathrm{ad}(g_{b,a})} & & \downarrow{\scriptstyle \mathrm{ad}(\Phi(g_{b,a}))} & & \downarrow{\scriptstyle \mathrm{ad}(h)} \\
\Gamma_b & \xrightarrow[\Phi_b]{} & \Gamma_{\phi(b)} & \xrightarrow[\mathrm{ad}(g_{b,\phi(b)})]{} & \Gamma_b,
\end{array}$$

where

$$h = g_{b,\phi(b)} \Phi(g_{b,a}) g_{a,\phi(a)}^{-1}. \tag{7}$$

Hence

$$\Phi_{(b)} \circ \mathrm{ad}(g_{b,a}) = \mathrm{ad}(h) \circ \Phi_{(a)}, \tag{8}$$

with h as in (7). Consequently,

$$\Phi_{(a)} \text{ is an inner automorphism iff } \Phi_{(b)} \text{ is an inner automorphism.} \tag{9}$$

Now using (4) and (5) we can define the map

$$\sigma'_a : \mathrm{Aut}(\mathfrak{A}) \to \mathrm{Aut}(\Gamma_a)_{l_a}, \quad \sigma'_a(\Phi) = \Phi_{(a)}. \tag{10}$$

However σ'_a is not quite a homomorphism. For let $\Phi' = (\phi', (\gamma')) \in \mathrm{Aut}(\mathfrak{A})$. Then

$$\sigma'_a(\Phi'\Phi) = \mathrm{ad}(g_{a,\phi'\phi(a)}) \circ (\Phi'\Phi)_a, \tag{11}$$

while, for $h \in \Gamma_a$, $g = g_{a,\phi(a)}$, and $g' = g_{a,\phi'(a)}$,

$$\begin{aligned}\sigma'_a(\Phi')\sigma'_a(\Phi)(h) &= g'(\Phi'_a(g\Phi_a(h)g^{-1}))g'^{-1} \\ &= g'\Phi'(g)\Phi'(\Phi_a(h))\Phi'(g)^{-1}g'^{-1} \\ &= (g'\Phi'(g))(\Phi'\Phi)_a(h)(g'\Phi'(g))^{-1}.\end{aligned}$$

Thus

$$\sigma'_a(\Phi')\sigma'_a(\Phi) = \mathrm{ad}(g_{a,\phi'(a)}\Phi'(g_{a,\phi(a)})) \circ (\Phi'\Phi)_a, \tag{12}$$

which differs from (11) by an inner automorphism

$$\mathrm{ad}(g_{a,\phi'(a)}\Phi'(g_{a,\phi(a)})g^{-1}_{a,\phi'\phi(a)}) \tag{13}$$

of Γ_a. Of course,

$$\text{On the group } \mathrm{Aut}^A(\mathfrak{A}) = \{\Phi \mid \phi_A = \mathrm{Id}_A\},\ \Phi_{(a)} = \Phi_a, \text{ and } \\ \sigma'_a : \mathrm{Aut}^A(\mathfrak{A}) \to \mathrm{Aut}(\Gamma_a)_{l_a} \text{ is a homomorphism.} \tag{14}$$

In general composing σ'_a with the projection $\mathrm{Aut}(\Gamma_a) \to \mathrm{Out}(\Gamma_a)$ thus defines a homomorphism

$$\sigma_a : \mathrm{Aut}(\mathfrak{A}) \to \mathrm{Out}(\Gamma_a)_{l_a}. \tag{15}$$

We define

$$\mathrm{In\,Aut}(\mathfrak{A}) = \mathrm{Ker}(\sigma_a). \tag{16}$$

This is, in view of (9), independent of a, and we define

$$\mathrm{Out}(\mathfrak{A}) = \mathrm{Aut}(\mathfrak{A})/\mathrm{In\,Aut}(\mathfrak{A}) \cong \mathrm{Im}(\sigma_a). \tag{17}$$

From (6) and Lemma 1.12 we see that,

$$\text{If } \mathfrak{A} \text{ is minimal non-abelian then } \mathrm{In\,Aut}(\mathfrak{A}) \le \mathrm{Aut}^A(\mathfrak{A}). \tag{18}$$

We shall see, in Corollary 4.2 below, that the homomorphism (15) is surjective, and so

$$\mathrm{Out}(\mathfrak{A}) \cong \mathrm{Out}(\Gamma_a)_{l_a}. \tag{19}$$

3.9. Morphisms induced on quotient graphs of groups (cf. [2, Section 4]). Let

$$(\alpha, \lambda) : (\Gamma, X) \to (\Gamma', X')$$

be a morphism of tree actions: $\lambda(gx) = \alpha(g)\lambda(x)$ for $g \in \Gamma$, $x \in X$. Suppose that we have constructed quotient graphs of groups

$$\Gamma\backslash\backslash X = \mathfrak{A} = (A, \mathscr{A}),$$

$$\Gamma'\backslash\backslash X' = \mathfrak{A}' = (A', \mathscr{A}')$$

as in 2.4. Then one can construct a morphism

$$\Phi = (\phi, (\gamma)) : \mathfrak{A} \to \mathfrak{A}'$$

with the following properties. The diagram

$$\begin{array}{ccc} \pi(\mathfrak{A}) & \xrightarrow{\ \Phi\ } & \pi(\mathfrak{A}') \\ {\scriptstyle \psi^{\mathfrak{A}}}\downarrow & & \downarrow{\scriptstyle \psi^{\mathfrak{A}'}} \\ \Gamma & \xrightarrow[\ \alpha\ ]{} & \Gamma' \end{array}$$

commutes, hence so also does

$$\begin{array}{ccc} \Gamma_a = \pi_1(\mathfrak{A}, a) & \xrightarrow{\ \Phi_a\ } & \Gamma'_{\phi(a)} = \pi_1(\mathfrak{A}', \phi(a)) \\ {\scriptstyle \psi_a^{\mathfrak{A}}}\downarrow & & \downarrow{\scriptstyle \psi_{\phi(a)}^{\mathfrak{A}'}} \\ \Gamma & \xrightarrow[\ \alpha\ ]{} & \Gamma' \end{array}$$

Further we have a commutative diagram

$$\begin{array}{ccc} X_a = (\widetilde{\mathfrak{A}, a}) & \xrightarrow{\ \bar{\Phi}_a\ } & X'_{\phi(a)} = (\widetilde{\mathfrak{A}', \phi(a)}) \\ {\scriptstyle \tau_a^{\mathfrak{A}}}\downarrow & & \downarrow{\scriptstyle \tau_{\phi(a)}^{\mathfrak{A}'}} \\ X & \xrightarrow[\ \lambda\ ]{} & X' \end{array}$$

Thus Φ "recovers" (α, λ) in the sense that it defines a commutative diagram of tree actions

$$\begin{array}{ccc} (\Gamma_a, X_a) & \xrightarrow{\ (\Phi_a, \bar{\Phi}_a)\ } & (\Gamma'_{\phi(a)}, X'_{\phi(a)}) \\ {\scriptstyle (\psi_a^{\mathfrak{A}}, \tau_a^{\mathfrak{A}})}\downarrow{\scriptstyle \cong} & & {\scriptstyle \cong}\downarrow{\scriptstyle (\psi_{\phi(a)}^{\mathfrak{A}'}, \tau_{\phi(a)}^{\mathfrak{A}'})} \\ (\Gamma, X) & \xrightarrow[\ (\alpha, \lambda)\ ]{} & (\Gamma', X') \end{array} \qquad (1)$$

Finally,

(α, λ) is an isomorphism iff Φ is an isomorphism.

4. Length preserving group automorphisms come from automorphisms of the quotient graph of groups

4.1. Theorem. *Let X be a minimal non-abelian Γ-tree, with hyperbolic length function $l = l_X$. Form a quotient graph of groups*

$$\Gamma\backslash\backslash X = \mathfrak{A} = (A, \mathscr{A}),$$

choose a base point $a_0 \in VA$, and use 2.4 to identify $\Gamma = \Gamma_{a_0} := \pi_1(\mathfrak{A}, a_0)$ and $X = X_{a_0} := (\widetilde{\mathfrak{A}, a_0})$. Let $\alpha \in \mathrm{Aut}(\Gamma)$. The following conditions are equivalent:
(a) $\alpha \in \mathrm{Aut}(\Gamma)_l : l(\alpha(g)) = l(g)$ for all $g \in \Gamma$.
(b) $\exists \Phi = (\phi, (\gamma)) \in \mathrm{Aut}(\mathfrak{A})$, and $h = |\omega|$, where ω is an edge path in A from a_0 to $\phi(a_0)$, such that $\alpha = \mathrm{ad}(h) \circ \Phi_{a_0}$.

$$\begin{array}{ccc} \Gamma = \pi_1(\mathfrak{A}, a_0) & \xrightarrow{\ \alpha\ } & \Gamma = \pi_1(\mathfrak{A}, a_0) \\ {\scriptstyle \Phi_{a_0}}\big\downarrow & & \big\uparrow{\scriptstyle \mathrm{ad}(h)} \\ \pi_1(\mathfrak{A}, \phi(a_0)) & = & \pi_1(\mathfrak{A}, \phi(a_0)) \end{array}$$

Proof. (b)$\Rightarrow$(a): This follows as in 3.6. Putting $\Gamma_a = \pi_1(\mathfrak{A}, a)$ and $X_a = (\widetilde{\mathfrak{A}, a})$, with length function l_a, we have isomorphisms of group actions

$$(\Gamma_a, X_a) \xrightarrow{(\Phi_a, \tilde{\Phi}_a)} (\Gamma_{\phi(a)}, X_{\phi(a)}) \xrightarrow{(\mathrm{ad}(h), h\cdot)} (\Gamma_a, X_a)$$

(cf. 3.3 and 2.3(3)). It follows then from Lemma 1.3 that $\mathrm{ad}(h) \circ \Phi_a$ preserves l_a.

(a)$\Rightarrow$(b): Suppose that $l \circ \alpha = l$ $(l = l_{a_0})$. Since X is a minimal non-abelian Γ-tree, it follows from Theorem 1.10 that there is a unique $\lambda \in \mathrm{Aut}(X)$ which is α-equivariant, i.e.

$$(\alpha, \lambda) : (\Gamma, X) \to (\Gamma, X_\alpha)$$

is an isomorphism of tree actions, where X_α denotes X with the given Γ-action composed with α. Now it follows from 2.5 that we can choose fundamental domain data so as to identify

$$\Gamma\backslash\backslash X_\alpha = \mathfrak{A} = \Gamma\backslash\backslash X.$$

Moreover the projection $\psi : \pi(\mathfrak{A}) \to \Gamma$ is the same for both interpretations of $\mathfrak{A}$.

Then (cf. 3.9(1)) the isomorphism (α,λ) permits us to construct $\Phi = (\phi,(\gamma)) \in \mathrm{Aut}(\mathfrak{A})$ such that we have a commutative diagram of isomorphisms

$$\begin{array}{ccc} (\Gamma_a, X_{a_0}) & \xrightarrow{(\Phi_{a_0},\bar{\Phi}_{a_0})} & (\Gamma_{\phi(a_0)}, X_{\phi(a_0)}) \\ {\scriptstyle (\psi_{a_0},\tau_{a_0})}\downarrow & & \downarrow{\scriptstyle (\psi_{\phi(a_0)},\tau'_{\phi(a_0)})} \\ (\Gamma, X) & \xrightarrow[(\alpha,\lambda)]{} & (\Gamma, X_a) \end{array}$$

Fix a spanning tree $T \subset A$ so that ψ factors through an isomorphism $\pi_1(\mathfrak{A},T) \to \Gamma$, which we view as an identification. For $a,b \in VA$ let $g_{a,b} \in \pi[a,b]$ come from the edge-path in T from a to b. Let $\sigma_a : \Gamma \to \Gamma_a$ denote the inverse of the isomorphism $\psi_a : \Gamma_a \to \Gamma$. Then the diagram above plus 2.2(13) furnish a commutative diagram

$$\begin{array}{ccccc} \Gamma_{a_0} & \xrightarrow{\Phi_{a_0}} & \Gamma_{\phi(a_0)} & \xrightarrow{\mathrm{ad}(g_{a_0,\phi(a_0)})} & \Gamma_{a_0} \\ {\scriptstyle \sigma_{a_0}}\uparrow & & \uparrow{\scriptstyle \sigma_{\phi(a_0)}} & & \uparrow{\scriptstyle \sigma_{a_0}} \\ \Gamma & \xrightarrow[\alpha]{} & \Gamma & = & \Gamma \end{array}$$

Thus, using σ_{a_0} to identify Γ with Γ_{a_0}, α is converted to $\mathrm{ad}(g_{a_0,\phi(a_0)}) \circ \Phi_{a_0}$, whence the theorem. □

4.2. Corollary. *Let $\Gamma, X, l = l_X$, and $\mathfrak{A} = \Gamma\backslash\backslash X$ be as in Theorem* 4.1. *Choose a base point $a_0 \in VA$ and identify (Γ, X) with (Γ_{a_0}, X_{a_0}). Then we have an exact sequence*

$$1 \to \mathrm{In\,Aut}(\mathfrak{A}) \to \mathrm{Aut}(\mathfrak{A}) \xrightarrow{\sigma_{a_0}} \mathrm{Out}(\Gamma)_l \to 1, \qquad (1)$$

where σ_{a_0} is as in 3.8(15).

Proof. The only non-trivial point is the surjectivity of σ_{a_0}, and this is given by Theorem 4.1, (a) $\Rightarrow$ (b). □

The sequence (1) permits us to use the study of $\mathrm{Aut}(\mathfrak{A})$, which we carry out in Sections 6 and 7, to obtain information about $\mathrm{Out}(\Gamma)_l$, described in Theorem 8.1.

In the next section, we apply Theorem 4.1 to the special case when $A = \Gamma\backslash X$ is an edge (amalgam) or a loop (HNN-extension).

5. Amalgams and HNN-extensions

5.1. Amalgams. Let $A = a \circ\!\xrightarrow{e}\!\circ\, b$, and view α_e and $\alpha_{\bar{e}}$ as inclusions of a proper subgroup,

$$\mathcal{A}_a \supsetneqq \mathcal{A}_e \subsetneqq \mathcal{A}_b. \qquad (1)$$

Then

$$\Gamma = \pi_1(\mathfrak{A}, A) = \mathscr{A}_a *_{\mathscr{A}_e} \mathscr{A}_b, \tag{2}$$

while

$$\Gamma_a = \pi_1(\mathfrak{A}, a) = \mathscr{A}_a *_{\mathscr{A}_e} e\mathscr{A}_b e^{-1} \leq \pi(\mathfrak{A}). \tag{3}$$

The map $\pi(\mathfrak{A}) \to \Gamma$ killing $e \in \pi(\mathfrak{A})$ induces an isomorphism $\Gamma_a \xrightarrow{\cong} \Gamma$. For $\gamma \in \mathrm{Aut}(\Gamma)$, let γ_a denote the corresponding automorphism of Γ_a.

Following Martindale and Montgomery [6] we call $\gamma \in \mathrm{Aut}(\Gamma)$ an *induced automorphism* if $\gamma(\mathscr{A}_c) = \mathscr{A}_c$ $(c = a, b)$, and an *exchange automorphism* if $\gamma(\mathscr{A}_a) = \mathscr{A}_b$ and $\gamma(\mathscr{A}_b) = \mathscr{A}_a$. Note that

$$\mathrm{Aut}(A) = \{I, \sigma\}, \qquad \sigma(e) = \bar{e}. \tag{4}$$

Let l denote the length function of the Γ-action on $X_a = (\widetilde{\mathfrak{A}, a})$.

5.2. Theorem. *Let $\gamma \in \mathrm{Aut}(\Gamma)$. Then $l \circ \gamma = l$ iff $\gamma = \mathrm{ad}(h) \circ \beta$, with $h \in \Gamma$ and β is either an induced or an exchange automorphism.*

Proof. We know from Theorem 4.1 that $l \circ \gamma = l$ iff $\gamma_a = \mathrm{ad}(g) \circ \Phi_a$, where $\Phi = (\phi, (\delta)) \in \delta\mathrm{Aut}(\mathfrak{A})$, and $g \in \pi[a, \phi_A(a)]$. Write

$$\Phi = (\phi_A, \{\phi_a, \phi_b\}, \{\phi_e\}, \{\delta_e, \delta_{\bar{e}}\}).$$

Then we can factor

$$\Phi = \Phi' \circ \Phi''$$

where

$$\Phi' = (\mathrm{Id}_A, \{\mathrm{ad}(\delta_e), \mathrm{ad}(\delta_{\bar{e}})\}, \{\mathrm{Id}_{\mathscr{A}_e}\}, \{\delta_e, \delta_{\bar{e}}\}),$$

$$\Phi'' = (\phi_A, \{\phi''_a, \phi''_b\}, \{\phi_e\}, \{1, 1\}),$$

$$\phi''_a = \mathrm{ad}(\delta_e^{-1}) \circ \phi_a, \qquad \phi''_b = \mathrm{ad}(\delta_{\bar{e}}^{-1}) \circ \phi_b.$$

An easy calculation verifies the above, as well as the fact that

$$\Phi'_a = \mathrm{ad}(\delta_e) : \Gamma_a \to \Gamma_a.$$

Thus, replacing g by $g\delta_e$, and Φ by Φ'', we reduce to the case when $\delta_e = 1 = \delta_{\bar{e}}$, which we now assume. It follows that, for $\Phi : \pi(\mathfrak{A}) \to \pi(\mathfrak{A})$, $\Phi(e) = \delta_e e \delta_{\bar{e}}^{-1} = e$. Thus

$$\Phi_a(\mathscr{A}_a) = \phi_a(\mathscr{A}_a) = \mathscr{A}_{\phi_A(a)},$$

and

$$\Phi_a(e\mathscr{A}_b e^{-1}) = e\phi_b(\mathscr{A}_b)e^{-1} = e\mathscr{A}_{\phi_A(b)}e^{-1}.$$

When $\phi_A = \mathrm{Id}_A$, Φ_a is induced. When $\phi_A = \sigma$, $g \in \pi[a,b]$, so $\gamma_a = \mathrm{ad}(ge^{-1}) \circ \mathrm{ad}(e) \circ \Phi_a$, with $ge^{-1} \in \Gamma_a$, and $\psi_a := \mathrm{ad}(e) \circ \Phi_a$ satisfies $\psi_a(\mathscr{A}_a) = e\phi_a(\mathscr{A}_a)e^{-1} = e\mathscr{A}_b e^{-1}$, while $\psi_a(e\mathscr{A}_b e^{-1}) = e(\sigma(e)\phi_b(\mathscr{A}_b)\sigma(e)^{-1})e^{-1} = e\bar{e}\mathscr{A}_a\bar{e}^{-1}e^{-1} = \mathscr{A}_a$. Thus ψ is an exchange automorphism.

To complete the proof it suffices to show conversely, that, if $\psi \in \mathrm{Aut}(\Gamma)$ is either induced or exchange, then $\psi_a = \mathrm{ad}(h) \circ \Phi_a$ for some $\Phi \in \delta\mathrm{Aut}(\mathfrak{A})$ and $h \in \pi[a, \phi_A(a)]$. Define $\phi_A = \mathrm{Id}_A$ if ψ is induced, and $\phi_A = \sigma$ if ψ is exchange. Let $\psi_c = \psi|_{\mathscr{A}_c} : \mathscr{A}_c \to \mathscr{A}_{\phi_A(c)}$ for $c = a, b$. Since, in Γ, $\mathscr{A}_e = \mathscr{A}_a \cap \mathscr{A}_b = \psi\mathscr{A}_a \cap \psi\mathscr{A}_b$, ψ induces an automorphism ψ_e of $\mathscr{A}_e$. Thus we have

$$\Phi = (\phi_A, \{\psi_a, \psi_b\}, \{\psi_e\}, \{1, 1\}) \in \delta\mathrm{Aut}(\mathfrak{A}).$$

It is easily calculated that $\psi_a = \Phi_a$ if ψ is induced, and $\psi_a = \mathrm{ad}(e) \circ \Phi_a$ if ψ is exchange. □

5.3. The stabilizer of l, $\mathrm{Aut}(\Gamma)_l$. For $c = a, b$, let $\mathrm{Aut}^E(\mathscr{A}_c)$ denote the stabilizer of $\mathscr{A}_e$ in $\mathrm{Aut}(\mathscr{A}_c)$. Then the restriction homomorphisms $\mathrm{Aut}^E(\mathscr{A}_c) \to \mathrm{Aut}(\mathscr{A}_e)$ allow us to define

$$\begin{aligned}\mathrm{IA} &= \mathrm{Aut}^E(\mathscr{A}_a) \times_{\mathrm{Aut}(\mathscr{A}_e)} \mathrm{Aut}^E(\mathscr{A}_b)\\ &= \{(\phi_a, \phi_b) \in \mathrm{Aut}(\mathscr{A}_a) \times \mathrm{Aut}(\mathscr{A}_b) \mid \phi_a|_{\mathscr{A}_e} = \phi_b|_{\mathscr{A}_e}\}.\end{aligned}$$

Clearly we can identify IA with the group of induced automorphisms of Γ. If there is an exchange automorphism γ, then γ^2 is induced, and $\langle \mathrm{IA}, \gamma \rangle$ is the group of induced and exchange automorphisms.

Put

$$\begin{aligned}N &= \{(\mathrm{ad}(g^{-1}), (\mathrm{ad}(g), \mathrm{ad}(g)) \mid g \in \mathscr{A}_e\}\\ &\le \mathrm{ad}(\Gamma) \rtimes \mathrm{IA}.\end{aligned}$$

5.4. Theorem. *If there is an exchange automorphism* γ *then*

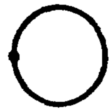
$$\mathrm{Aut}(\Gamma)_l \cong (\mathrm{ad}(\Gamma) \rtimes \langle \mathrm{IA}, \gamma \rangle)/N,$$

otherwise

$$\mathrm{Aut}(\Gamma)_l \cong (\mathrm{ad}(\Gamma) \rtimes \mathrm{IA})/N.$$

Proof. See Lemma 5.2 of [5]. □

5.5. HNN-extensions. Let

$A = a$  $e.$

$$\mathscr{A}_e \underset{\alpha_{\bar{e}}}{\overset{\alpha_e}{\rightrightarrows}} \mathscr{A}_a \tag{1}$$

Then

$$\Gamma = \pi_1(\mathfrak{A}, a) = \langle \mathscr{A}_a, e \mid e\alpha_{\bar{e}}(s)e^{-1} = \alpha_e(s)\ \forall s \in \mathscr{A}_e \rangle \tag{2}$$

is the HNN-extension associated with (1). Note that,

$$\mathrm{Aut}(A) = \{\mathrm{I}, \sigma\}, \qquad \sigma(e) = \bar{e}. \tag{3}$$

Let l denote the length function of the Γ-action on $(\widetilde{\mathfrak{A}, a})$.

5.6. Theorem. *Let $\gamma \in \mathrm{Aut}(\Gamma)$. Then $l \circ \gamma = l$ iff $\gamma = \mathrm{ad}(g) \circ \psi$ with $g \in \Gamma$ and ψ of one of the following forms*:

(1) $\psi(\mathscr{A}_a) = \mathscr{A}_a$, $\psi(\alpha_{\bar{e}}\mathscr{A}_e) = \alpha_{\bar{e}}\mathscr{A}_e$, $\psi(e) = \delta_e e$, $\delta_e \in \mathscr{A}_a$, *and* $\mathrm{ad}(\delta_e e) \circ \psi \circ \alpha_e = \psi \circ \alpha_{\bar{e}}$.

(2) $\psi(\mathscr{A}_a) = \mathscr{A}_a$, $\psi(\alpha_{\bar{e}}\mathscr{A}_e) = \alpha_e\mathscr{A}_e$, $\psi(e) = \delta_e e^{-1}$, $\delta_e \in \mathscr{A}_a$, *and* $\mathrm{ad}(\delta_e e^{-1}) \circ \psi \circ \alpha_{\bar{e}} = \psi \circ \alpha_e$.

Proof. From Theorem 4.1 we know that $l \circ \gamma = l$ iff $\gamma = \mathrm{ad}(g) \circ \Phi_a$ for some $\Phi \in \delta\,\mathrm{Aut}(\mathfrak{A})$ and $g \in \Gamma$. Writing

$$\Phi = (\phi_A, \{\phi_a\}, \{\phi_e\}, \{\delta_e, \delta_{\bar{e}}\})$$

we can factor

$$\Phi = \Phi' \circ \Phi'',$$

where

$$\Phi' = (\mathrm{Id}_A, \{\mathrm{ad}(\delta_{\bar{e}})\}, \{\mathrm{Id}_{\mathscr{A}_e}\}, \{\delta_{\bar{e}}, \delta_{\bar{e}}\}),$$

$$\Phi'' = (\phi_A, \{\mathrm{ad}(\delta_{\bar{e}}^{-1}) \circ \phi_a\}, \{\phi_e\}, \{\delta_{\bar{e}}^{-1}\delta_e, 1\}).$$

An easy calculation shows that $\Phi'_a = \mathrm{ad}(\delta_{\bar{e}}) : \Gamma \to \Gamma$. Thus, replacing g by $g\delta_{\bar{e}}$ and Φ by Φ'', we can reduce to the case $\delta_{\bar{e}} = 1$, which we now assume. Then we have commutative diagrams

$$(e):\quad \begin{array}{ccc} \mathscr{A}_a & \xrightarrow{\operatorname{ad}(\delta_e^{-1})\circ\phi_a} & \mathscr{A}_a \\ {\scriptstyle \alpha_e}\uparrow & & \uparrow{\scriptstyle \alpha_{\phi_A(e)}} \\ \mathscr{A}_e & \xrightarrow[\phi_e]{} & \mathscr{A}_e \end{array} \qquad (\bar{e}):\quad \begin{array}{ccc} \mathscr{A}_a & \xrightarrow{\phi_a} & \mathscr{A}_a \\ {\scriptstyle \alpha_{\bar{e}}}\uparrow & & \uparrow{\scriptstyle \alpha_{\phi_A(\bar{e})}} \\ \mathscr{A}_e & \xrightarrow[\phi_e]{} & \mathscr{A}_e \end{array} \tag{1}$$

From the diagram $(1)(\bar{e})$ we see that

$$\phi_a(\alpha_{\bar{e}}\mathscr{A}_e) = \alpha_{\phi_A(\bar{e})}\mathscr{A}_e. \tag{2}$$

Further

$$\operatorname{ad}(\delta_e^{-1}) \circ \phi_a \circ \alpha_e = \alpha_{\phi_A(e)} \circ \phi_e \quad \text{and} \quad \phi_a \circ \alpha_{\bar{e}} = \alpha_{\phi_A(\bar{e})} \circ \phi_e. \tag{3}$$

Let $\psi = \Phi_a : \Gamma \to \Gamma$. Then

$$\begin{aligned} &\psi|_{\mathscr{A}_a} = \phi_a : \mathscr{A}_a \to \mathscr{A}_a, \\ &\psi(\alpha_{\bar{e}}\mathscr{A}_e) = \alpha_{\phi_A(\bar{e})}\mathscr{A}_e, \\ &\psi(e) = \delta_e\phi_A(e). \end{aligned} \tag{4}$$

Case $\phi_A = \mathrm{Id}_A$. Then $\psi(\alpha_{\bar{e}}\mathscr{A}_e) = \alpha_{\bar{e}}\mathscr{A}_e$, $\psi(e) = \delta_e e$, and (cf. (3)) $\operatorname{ad}(\delta_e^{-1})\phi_a\alpha_e = \alpha_e\phi_e = \operatorname{ad}(e)\alpha_{\bar{e}}\phi_e = \operatorname{ad}(e)\phi_a\alpha_{\bar{e}}$, so

$$\operatorname{ad}(\delta_e e)\phi_a\alpha_e = \phi_a\alpha_{\bar{e}} \tag{5}$$

Case $\phi_A = \sigma$. Then $\psi(\alpha_{\bar{e}}\mathscr{A}_e) = \alpha_e\mathscr{A}_e$, $\psi(e) = \delta_e e^{-1}$, and (cf. (3)) $\operatorname{ad}(\delta_e^{-1})\phi_a\alpha_e = \alpha_{\bar{e}}\phi_e = \operatorname{ad}(e^{-1})\alpha_e\phi_e = \operatorname{ad}(e^{-1})\phi_a\alpha_{\bar{e}}$, so

$$\operatorname{ad}(\delta_e e^{-1})\phi_a\alpha_{\bar{e}} = \phi_a\alpha_e. \tag{6}$$

Conversely, let $\psi \in \operatorname{Aut}(\Gamma)$ satisfy (1) or (2). Then we can define $\phi_a \in \operatorname{Aut}(\mathscr{A}_a)$ and $\phi_e = \phi_{\bar{e}} \in \operatorname{Aut}(\mathscr{A}_e)$ by $\phi_a = \psi|_{\mathscr{A}_a}$, and $\phi_a \circ \alpha_{\bar{e}} = \alpha_{\phi_A(\bar{e})} \circ \phi_e$, where $\phi_A = \mathrm{Id}_A$ in case (1), and σ in case (2). The latter gives the commutative diagram $(1)(\bar{e})$. The commutativity of $(1)(e)$ follows from the hypothesis (5) in case (1), and (6) in case (2). Thus we have $\psi = \Phi_a$, where

$$\Phi = (\phi_A, \{\phi_a\}, \{\phi_e\}, \{\delta_e, 1\}).$$

Let F_n be a free group of rank n. Suppose that F_n acts freely (without inversions) and minimally on a tree X with a hyperbolic length function l.

5.7. Proposition. *Let* $A = F_n\backslash X$. *Let* $\varphi \in \operatorname{Aut}(F_n)$. *Then* $l \circ \varphi = l$ *iff there is an isomorphism* $\phi : A \to A$, *and an edge path* γ *from* a_0 *to* $\phi(a_0)$ *such that*

$$\varphi(e_1e_2\cdots e_n) = \gamma\phi(e_1)\cdots\phi(e_n)\gamma^{-1}$$

for all edge loop $e_1e_2\cdots e_n \in \pi_1(A, a_0)$.

In particular if A consists of one vertex and n geometric edges $\{e_1, e_2, \ldots, e_n\}$ *(A is a "rose"), then* $l \circ \varphi = l$ *iff there is a* $\gamma \in F_n$ *and a permutation* $\sigma \in S_n$ *such that* $\varphi(e_i) = \gamma e_{\sigma(i)}^{\pm 1} \gamma^{-1}$ $(1 \le i \le n)$.

Proof. The proof is left to the reader. □

5.8. Bounded automorphisms. Let $\Gamma = \pi_1(\mathfrak{A}, a)$ act on $X = (\widetilde{\mathfrak{A}, a})$ with hyperbolic length function l. Let $x_0 = [1]_a \in X$. Then for $L = L_{\mathfrak{A}}$ the path length function on Γ defined as in 2.2(5), it follows from 2.3(1) that

$$L(g) = d_X(gx_0, x_0) \quad \forall g \in \Gamma. \tag{1}$$

It then follows further from [1], that

$$l(g) = \mathrm{Max}(L(g^2) - L(g), 0) \quad \forall g \in \Gamma. \tag{2}$$

Let $H \subset \Gamma$ be a subset stable under squaring. It follows then from (2) that if $L(H)$ is bounded then $l(H)$ is bounded. If H is a subgroup then $l(H)$ can be bounded only if $l(H) = \{0\}$; indeed $l(g^n) = |n| l(g)$ for $g \in \Gamma$ and $n \in \mathbf{Z}$.

If, conversely, $l(H) = 0$ for $H \le \Gamma$, then either (i) H fixes some $x \in VX$, or (ii) H fixes an end ε of X, but no vertex (cf. [2, (7.2)]). In case (i), H is contained in a conjugate of some $\mathscr{A}_b$, and so $L(H)$ is bounded. However, in case (ii), $L(H)$ will not be bounded.

Call a *subgroup* $H \le \Gamma$ *bounded* if $L(H)$ is bounded. Call an *automorphism* $\alpha \in \mathrm{Aut}(\Gamma)$ *bounded* if $\alpha(H)$ is bounded for all bounded $H \le \Gamma$. If α is bounded then it follows from the discussion above that, for all $x \in X$, $\alpha(\Gamma_x) \le \Gamma_y$ for some $y \in X$. In fact, if α and α^{-1} are bounded, then α permutes the maximal bounded subgroups (= maximal vertex stabilizers) of Γ, and so, if Γ_x is a maximal vertex stabilizer, then $\alpha(\Gamma_x) = \Gamma_y$ for some $y \in X$.

5.9. Corollary. *Let* $\alpha \in \mathrm{Aut}(\Gamma)$. *If* $l \circ \alpha = l$ *then* α *and* α^{-1} *are bounded.*

Proof. Since $l \circ \alpha^{-1} = l$ it suffices to treat α. By Theorem 4.1, $\alpha = \Phi_{(a)} = \mathrm{ad}(\gamma) \circ \delta\Phi_a$ for some $\gamma \in \pi(\mathfrak{A})$ and $\delta\Phi \in \delta\,\mathrm{Aut}(\mathfrak{A})$. By Lemma 3.5, $\delta\Phi_a$ preserves L, and clearly $\mathrm{ad}(\gamma)$ increases L by at most an additive constant $(2 \cdot L(\gamma))$. □

6. The structure of Aut($\mathfrak{A}$) and In Aut($\mathfrak{A}$)

6.0. Composition and the center $\mathbf{Z}(\mathfrak{A})$. In this section we fix a graph of groups $\mathfrak{A} = (A, \mathscr{A})$, and put

$$\mathbf{G} = \mathrm{Aut}(\mathfrak{A}). \tag{1}$$

For $a \in VA$ we write $\Gamma_a = \pi_1(\mathfrak{A}, a)$ and $X_a = (\widetilde{\mathfrak{A}, a})$.

For reference, we recall the composition

$$\Phi'' = ((\phi'', (\gamma'')) = \Phi' \circ \Phi \tag{2}$$

of $\Phi = (\phi, (\gamma))$ with $\Phi' = (\phi', (\gamma'))$ (cf. (3.6), and [2, (2.11)]).

$$\phi''_A = \phi'_A \circ \phi_A. \tag{3}$$

For $e \in EA$, $\partial_0 = a \in VA$,

$$\phi''_a = \phi'_{\phi(a)} \circ \phi_a, \qquad \phi''_e = \phi'_{\phi(e)} \circ \phi_e, \tag{4}$$

$$\gamma''_a = \Phi'_{\phi(a)}(\gamma_a)\gamma'_{\phi(a)}, \qquad \gamma''_e = \Phi'_{\phi(a)}(\gamma_e)\gamma'_{\phi(e)}. \tag{5}$$

With $\delta_e = \gamma_a^{-1}\gamma_e$, $\delta'_e = \gamma'^{-1}_a\gamma'_e$, and $\delta''_e = \gamma''^{-1}_a\gamma''_e$, this gives

$$\delta''_e = \phi'_{\phi(a)}(\delta_e)\delta'_{\phi(e)}. \tag{6}$$

In some places we shall make use of the following hypothesis.

(MNA) The Γ_a-tree X_a is minimal non-abelian.

This condition is independent of a, so we can say similarly,

(MNA) "$\mathfrak{A}$ is minimal non-abelian."

In this case it follows from Proposition 1.5 that the center

$$Z_a(\mathfrak{A}) \coloneqq Z(\Gamma_a) \tag{7}$$

acts trivially on X_a, and so $Z_a(\mathfrak{A}) \le \mathscr{A}_a$, in fact

$$Z_a(\mathfrak{A}) \le \alpha_e(\mathscr{A}_e) \quad \forall e \in E_0(a). \tag{8}$$

Let $z_a \in Z_a(\mathfrak{A})$. For $b \in VA$ define $z_b = gz_ag^{-1}$, where $g \in \pi[b,a]$. Since $\pi[b,a] = g\Gamma_a$, this definition is independent of the choice of g. Moreover if $h \in \pi[c,b]$ then $hz_bh^{-1} = h_c$. Putting

$$z = (z_b)_{b\in VA}, \tag{9}$$

we see that such elements z form a group

$$Z(\mathfrak{A})$$

such that

$$Z(\mathfrak{A}) \xrightarrow{\cong} Z_b(\mathfrak{A}), \qquad z \mapsto z_b, \tag{10}$$

is an isomorphism for all $b \in VA$. We call $Z(\mathfrak{A})$ the "*center of* $\mathfrak{A}$".

Let $a \in VA$ and $e \in E_0(a)$. It follows from (8) that we can define $Z_e(\mathfrak{A}) \leq \mathscr{A}_e$ by

$$Z_a(\mathfrak{A}) = \alpha_e Z_e(\mathfrak{A}). \tag{11}$$

For $z \in Z(\mathfrak{A})$ we have

$$z_a = \alpha_e(z_e) \quad \text{for a unique } z_e \in Z_e(\mathfrak{A}). \tag{12}$$

If $\partial_1 e = b$ then

$$\alpha_e(z_{\bar{e}}) = e\alpha_{\bar{e}}(z_{\bar{e}})e^{-1} = ez_b e^{-1} = z_a = \alpha_e(z_e),$$

whence

$$z_{\bar{e}} = z_e. \tag{13}$$

6.1. The group $\mathbf{G}^A = \operatorname{Ker}(q_A)$, in the exact sequence from 3.7,

$$1 \to \mathbf{G}^A \to \mathbf{G} \xrightarrow{q_A} \operatorname{Aut}(A), \tag{1}$$

where

$$q_A(\Phi) = \phi_A, \qquad \mathbf{G}^A = \{\Phi | \phi_A = \mathrm{Id}_A\}. \tag{2}$$

We then further have the homomorphism

$$\mathbf{G}^A \xrightarrow{q} \prod_{a \in VA} \operatorname{Aut}(\mathscr{A}_a) \times \prod_{e \in EA} \operatorname{Aut}(\mathscr{A}_e),$$

$$q(\Phi) = ((\phi_a)_{a \in VA}, \ (\phi_e)_{e \in EA}).$$

This permits us to define normal subgroups

$$\mathbf{G}^{(V,E)} \lhd \mathbf{G}^{(V)} \lhd \mathbf{G}^A \tag{3}$$

where, for $\Phi = (\phi, (\gamma)) \in \mathbf{G}^A$

$$\Phi \in \mathbf{G}^{(\mathrm{V})} \text{ iff } \phi_a \in \operatorname{ad}(\mathscr{A}_a) \ \forall a \in VA, \tag{4}$$

and

$$\Phi \in \mathbf{G}^{(V,E)} \text{ iff } \phi_u \in \operatorname{ad}(\mathscr{A}_u) \ \forall u \in VA \cup EA. \tag{5}$$

6.2. The group $\operatorname{In} \mathbf{G} := \operatorname{In} \operatorname{Aut}(\mathfrak{A})$. Recall from 3.8(16) that this is the kernel of σ_a in the commutative diagram

$$\begin{array}{ccc} G & \xrightarrow{\sigma'_a} & \operatorname{Aut}(\Gamma_a) \\ \Vert & & \downarrow \text{proj} \\ G & \xrightarrow[\sigma_a]{} & \operatorname{Out}(\Gamma_a), \end{array}$$

where $\sigma'_a(\Phi) = \Phi_{(a)}$, as in 3.8(3). Thus

$$\mathrm{In}\,\mathbf{G} = \{\Phi \mid \Phi_{(a)} \in \mathrm{ad}(\Gamma_a) = \mathrm{In}\,\mathrm{Aut}(\Gamma_a)\}, \tag{1}$$

and this definition is independent of $a \in VA$.

We now make the assumption

(MNA) $\mathfrak{A}$ is minimal non-abelian.

It follows then from 3.8(18) that

$$\mathrm{In}\,\mathbf{G} \le \mathbf{G}^A, \tag{2}$$

and from Corollary 4.2 that we have an exact sequence

$$1 \to \mathrm{In}\,\mathbf{G} \to \mathbf{G} \xrightarrow{\sigma_a} \mathrm{Out}(\Gamma_a)_{I_a} \to 1. \tag{3}$$

To analyze $\mathrm{Out}(\Gamma_a)_{I_a}$ we shall introduce a chain of normal subgroups between $\mathbf{G}$ and $\mathrm{In}\,\mathbf{G}$.

6.3. The homomorphism $\delta : \mathbf{G} \to \mathbf{G}$ is defined on $\Phi = (\phi, (\gamma))$, as in 3.4, by $\delta\Phi = (\phi, (\delta))$, where ϕ is left unaltered, γ_a is replaced by $\delta_a = 1$, and γ_e is replaced by $\delta_e = \gamma_a^{-1}\gamma_e$. We have 3.4(1),

$$\Phi_a = \mathrm{ad}(\gamma_a) \circ (\delta\Phi)_a : \Gamma_a \to \Gamma_{\phi(a)}. \tag{1}$$

From 3.6(6) we know that δ is a homomorphism,

$$\delta(\Phi' \circ \Phi) = \delta\Phi' \circ \delta\Phi. \tag{2}$$

Further (cf. 3.4(2)) δ is clearly idempotent,

$$\delta^2 = \delta. \tag{3}$$

Thus

$$\mathbf{G} = \gamma\mathbf{G} \rtimes \delta\mathbf{G}, \quad \text{where}$$
$$\gamma\mathbf{G} = \mathrm{Ker}(\delta). \tag{4}$$

It is easily seen that we have an isomorphism

$$\prod_{a \in VA} \Gamma_a \xrightarrow{\cong} \gamma\mathbf{G} \tag{5}$$

sending $g = (g_a)_{a \in \mathrm{VA}}$ to $\Phi_g = (I, (\gamma))$, defined by $I_A = \mathrm{Id}_A$, $I_u = \mathrm{Id}_{\mathscr{A}_u}$ for $u \in VA \cup EA$, and, for $a \in VA$, $e \in E_0(a)$, $\gamma_a = g_a = \gamma_e$ (whence $\delta_e = 1$). Since $(\Phi_g)_a = \mathrm{ad}(g_a)$, clearly, we have

$$\gamma\mathbf{G} \le \mathrm{In}\,\mathbf{G}. \tag{6}$$

It follows that,

$$\text{If} \quad \operatorname{In}\mathbf{G} \le H \le \mathbf{G}, \quad \text{then } H = \gamma\mathbf{G} \rtimes \delta H, \tag{7}$$

and

$$\sigma_a H = \sigma_a \delta H, \quad \text{where } \sigma_a : \mathbf{G} \to \operatorname{Out}(\Gamma_a)_{\iota_a}. \tag{8}$$

6.4. Theorem. *Continue to assume (MNA) : $\mathfrak{A}$ is minimal non-abelian. Let $\Phi = (\phi, (\delta)) \in \delta\mathbf{G}$. Then $\Phi \in \operatorname{In}\mathbf{G}$ iff the following conditions hold:*
(a) *$\phi_A = \mathrm{Id}_A$, i.e. $\Phi \in \delta\mathbf{G}^A$.*
(b) *There exist elements $h_a \in \mathscr{A}_a$ $(a \in VA)$ and $s_e \in \mathscr{A}_e$ $(e \in EA)$ such that,*

$$\phi_a = \operatorname{ad}(h_a), \qquad \phi_e = \operatorname{ad}(s_e) \quad \text{and} \quad \delta_e = h_a\alpha_e(s_e)^{-1} \quad \text{if } \partial_0 e = a.$$

(c) *For all $e \in EA$, the element $z_e(e) := s_e^{-1}s_{\overline{e}}$ belongs to $Z_e(\mathfrak{A})$ (cf. 6.1). This defines an element $z(e) \in Z(\mathfrak{A})$.*
(d) *For each closed path $(e_1, \ldots, e_n)$ in A,*

$$z(e_1)\cdots z(e_n) = 1.$$

Under these conditions, $\Phi_a = \operatorname{ad}(h_a) : \Gamma_a \to \Gamma_a$.

Proof. First assume that $\Phi \in \delta\operatorname{In}\mathbf{G}$. Then (a) follows from 6.2(2). By assumption, for each $a \in VA$, there is an $h_a \in \Gamma_a$ such that $\Phi_a (= \Phi_{(a)}) = \operatorname{ad}(h_a)$.

$$\Phi_a (= \Phi_{(a)}) = \operatorname{ad}(h_a) : \Gamma_a \to \Gamma_a. \tag{1}$$

Let $g \in \Gamma_a$, $e \in E_0(a)$, and $b = \partial_1 e$. Then $e^{-1}ge \in \Gamma_b$, so

$$\begin{aligned} h_b(e^{-1}ge)h_b^{-1} &= \Phi_b(e^{-1}ge) \\ &= (\delta_e e \delta_{\overline{e}}^{-1})^{-1}(h_a g h_a^{-1})(\delta_e e \delta_{\overline{e}}^{-1}) \\ &= (h_a^{-1}\delta_e e \delta_{\overline{e}}^{-1})^{-1} g (h_a^{-1}\delta_e e \delta_{\overline{e}}^{-1}). \end{aligned}$$

Hence

$$z_a(e) := h_a^{-1}\delta_e e \delta_{\overline{e}}^{-1} h_b e^{-1} \in Z_a(\mathfrak{A}) \quad (= Z(\Gamma_a)), \tag{2}$$

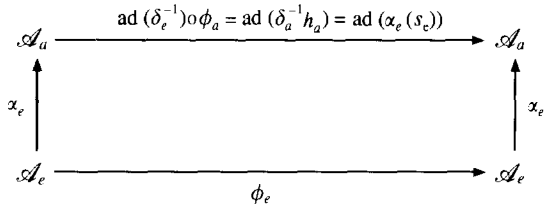
since $z_a(e)$ commutes with all $g \in \Gamma_a$. As in 6.0(9), this defines an element

$$z(e) \in Z(\mathfrak{A}).$$

Now $z_a(e) = |(h_a^{-1}\delta_e,\ e,\ \delta_{\overline{e}}^{-1}h_b,\ \overline{e})| \in \mathscr{A}_a$. Hence the indicated path cannot be reduced (cf. 2.6). It follows that $\delta_{\overline{e}}^{-1}h_b = \alpha_{\overline{e}}(s_{\overline{e}})$ for some $s_{\overline{e}} \in \mathscr{A}_{\overline{e}}$, and so $h_b = \delta_{\overline{e}}\alpha_{\overline{e}}(s_{\overline{e}})$. Applied to $\overline{e}$ in place of e, we obtain

$$h_a = \delta_e\alpha_e(s_e) \quad \text{for some } s_e \in \mathscr{A}_e \tag{3}$$

for each $a \in VA$, $e \in E_0(a)$. From (1), (3), and the commutative diagram



we see that

$$\phi_e = \mathrm{ad}(s_e), \tag{4}$$

whence condition (b). From (2) and (3),

$$\begin{aligned} z_a(e) &= \alpha_e(s_e^{-1}) e \alpha_{\bar{e}}(s_{\bar{e}}) e^{-1} \\ &= \alpha_e(s_e^{-1}) \alpha_e(s_{\bar{e}}) \\ &= \alpha_e(s_e^{-1} s_{\bar{e}}) \in Z_a(\mathfrak{A}), \end{aligned} \tag{5}$$

i.e.

$$z_e(e) = s_e^{-1} s_{\bar{e}} \in Z_e(\mathfrak{A}), \tag{6}$$

whence condition (c). Next note that, if $\partial_0 e = a$, $\partial_1 e = b$, we have

$$\begin{aligned} \Phi(e) = \delta_e e \delta_{\bar{e}}^{-1} &\overset{(3)}{=} (h_a \alpha_e(s_e)^{-1}) e (h_b \alpha_{\bar{e}}(s_{\bar{e}})^{-1})^{-1} \\ &= h_a \alpha_e(s_e)^{-1} e \alpha_{\bar{e}}(s_{\bar{e}}) h_b^{-1} \\ &= h_a (\alpha_e(s_e)^{-1} e \alpha_{\bar{e}}(s_{\bar{e}}) e^{-1})(e h_b^{-1}) \\ &\overset{(5)}{=} h_a z_a(e) e h_b^{-1} \overset{(5)}{=} z_a(e) h_a e h_b^{-1}. \end{aligned} \tag{7}$$

Now for any path $\gamma = (g_0, e_1, g_1, \ldots, e_n, g_n)$ in $\mathfrak{A}$, say from $a = \partial_0 e$ to $b = \partial_1 e_n$, define

$$z(\gamma) = z(e_1) \cdots z(e_n) \in Z(\mathfrak{A}) \qquad \text{(cf. 6.0(9))}. \tag{8}$$

Then it follows inductively from (1) and (7) that

$$\Phi(|\gamma|) = z_a(\gamma) h_a |\gamma| h_b^{-1}. \tag{9}$$

When γ is a closed path ($b = a$), it follows from (1), (8), and (9) that

$$z(e_1) \cdots z(e_n) = 1 \quad \text{for all closed paths } (e_1, \ldots, e_n) \text{ in } A, \tag{10}$$

whence condition (d).

Now, conversely, suppose that Φ satisfies (a)–(d). Then we have elements $h_a \in \mathscr{A}_a$, $s_e \in \mathscr{A}_e$, $z_e(e) = s_e^{-1} s_{\bar{e}} \in Z_e(\mathfrak{A})$, and we have the relations

$$\phi_a = \mathrm{ad}(h_a) : \mathscr{A}_a \to \mathscr{A}_a \tag{11}$$

as well as (2)–(5). It follows that the calculation (7) remains valid, and hence so also the relations (8) and (9). From (9) it follows that

$$\Phi_a(g) = z_a(g)h_a g h_a^{-1} \quad (g \in \Gamma_a) \tag{12}$$

where $z_a : \Gamma_a \to Z_a(\mathfrak{A})$ is the homomorphism defined by (8), via the natural projection $\Gamma_a = \pi_1(\mathfrak{A}, a) \to \pi_1(A, a)$. Finally, condition (d) says that the homomorphism z_a is trivial, and so $\Phi_a = \mathrm{ad}(h_a)$, whence $\Phi \in \mathrm{In}\,\mathbf{G}$, as claimed. □

6.5. Corollary. *With the notation of* 6.1, *we have*

$$\mathrm{In}\,\mathbf{G} \lhd \mathbf{G}^{(V,E)} \lhd \mathbf{G}^{(V)} \lhd \mathbf{G}^{A} \lhd \mathbf{G}$$

6.6. Successive quotients. Recall the surjection 3.8(10)

$$\sigma_a' : \mathbf{G} \twoheadrightarrow \mathrm{Aut}(\Gamma_a)_{l_a} \tag{1}$$

which projects to the homomorphism σ_a in the exact sequence of Corollary 4.2

$$1 \to \mathrm{In}\,\mathbf{G} \to \mathbf{G} \xrightarrow{\sigma_a} \mathrm{Out}(\Gamma_a)_{l_a} \to 1. \tag{2}$$

The restriction of σ_a',

$$\sigma_a' : \mathbf{G}^A \to \mathrm{Aut}(\Gamma_a)_{l_a} \tag{3}$$

is a homomorphism 3.8(14). For each superscript $X = A, (V)$, or (V, E) above, we shall write $\mathrm{Aut}(\Gamma_a)_{l_a}^X = \sigma_a' \mathbf{G}^X$, and $\mathrm{Out}(\Gamma_a)_{l_a}^X = \sigma_a \mathbf{G}^X = \sigma_a \delta \mathbf{G}^X$. Thus we have

$$\mathrm{Out}(\Gamma_a)_{l_a}^{(V,E)} \lhd \mathrm{Out}(\Gamma_a)_{l_a}^{(V)} \lhd \mathrm{Out}(\Gamma_a)_{l_a}^{A} \lhd \mathrm{Out}(\Gamma_a)_{l_a}, \tag{4}$$

with successive quotients isomorphic to the corresponding quotients of $\mathbf{G}$ or of $\delta\mathbf{G}$.

We begin by observing that

$$\mathbf{G}/\mathbf{G}^A = \delta\mathbf{G}/\delta\mathbf{G}^A \cong \mathrm{Out}(\Gamma_a)_{l_a}/\mathrm{Out}(\Gamma_a)_{l_a}^A \leq \mathrm{Aut}(A), \tag{5}$$

where $\mathrm{Aut}(A)$ denotes the group of graph automorphisms of A. In many cases of interest, e.g. when Γ_a is finitely generated, the graph A is finite [2, (7.9)], and hence so also is the group $\mathrm{Aut}(A)$.

6.7. The groups $\mathbf{Aut}^E(\mathscr{A}_a)$ and the quotient $\mathbf{G}^A/\mathbf{G}^{(V)}$. For $a \in VA$, define

$$\mathrm{Aut}^E(\mathscr{A}_a) = \left\{ \phi \in \mathrm{Aut}(\mathscr{A}_a) \;\middle|\; \begin{array}{l} \phi\alpha_e\mathscr{A}_e \text{ is } \mathscr{A}_a\text{-conjugate} \\ \text{to } \alpha_e\mathscr{A}_e \ \forall e \in E_0(a) \end{array} \right\} \tag{1}$$

and

$$\mathrm{Out}^E(\mathscr{A}_a) = \mathrm{Aut}^E(\mathscr{A}_a)/\mathrm{ad}(\mathscr{A}_a). \tag{2}$$

Let $\Phi = (\phi, (\gamma)) \in \mathbf{G}^A$, and $\delta\Phi = (\phi, (\delta)) \in \delta\mathbf{G}^A$. The commutative diagram (for $a \in VA$, $e \in E_0(a)$),

$$\begin{array}{ccc} \mathscr{A}_a & \xrightarrow{\operatorname{ad}(\delta_e^{-1}) \circ \phi_a} & \mathscr{A}_a \\ \Big\uparrow{\scriptstyle \alpha_e} & & \Big\uparrow{\scriptstyle \alpha_e} \\ \mathscr{A}_c & \xrightarrow[\phi_e]{} & \mathscr{A}_c \end{array} \tag{3}$$

shows that

$$\phi_a \in \operatorname{Aut}^E(\mathscr{A}_a), \tag{4}$$

and further that

$$\phi_e \in \operatorname{Aut}(\mathscr{A}_e) \quad \text{extends, via } \alpha_e, \text{ to an automorphism in } \operatorname{Aut}^E(\mathscr{A}_a). \tag{5}$$

From (4) we have a homomorphism

$$\begin{aligned} &\phi_V : \mathbf{G}^A \to \prod_{a \in VA} \operatorname{Aut}^E(\mathscr{A}_a), \\ &\phi_V(\Phi) = (\phi_a)_{a \in VA} = \phi_V(\delta\Phi), \end{aligned} \tag{6}$$

and also

$$\begin{aligned} &\phi_{(V)} : \mathbf{G}^A / \operatorname{In} \mathbf{G} \to \prod_{a \in VA} \operatorname{Out}^E(\mathscr{A}_a), \\ &\operatorname{Ker}(\phi_{(V)}) = \mathbf{G}^{(V)}. \end{aligned} \tag{7}$$

Concerning the image of ϕ_V, consider an element

$$(\phi_a)_{a \in VA} \in \prod_{a \in VA} \operatorname{Aut}^E(\mathscr{A}_a). \tag{8}$$

By (1), there exist elements $\delta_e \in \mathscr{A}_a$ ($e \in E_0(a)$) such that $\operatorname{ad}(\delta_e^{-1}) \circ \phi_a$ stabilizes $\alpha_e \mathscr{A}_e$, and hence induces a $\phi_e \in \operatorname{Aut}(\mathscr{A}_e)$ such that diagram (3) commutes. Then, with $\phi_A = \operatorname{Id}_A$, we have defined a candidate $\Phi = (\phi, (\delta))$ with $\phi_V(\Phi) = (\phi_a)_{a \in VA}$. The only remaining obstacle is that, for Φ to belong to $\mathbf{G}$, we must have

$$\phi_e = \phi_{\bar{e}} \quad \forall e \in EA. \tag{9}$$

Thus, if $\partial_0 e = a$ and $\partial_1 e = b$, we require an automorphism ε of $\mathscr{A}_e$ making the following diagram commute.

$$\begin{array}{ccccc} \mathscr{A}_a & \xleftarrow{\alpha_e} & \mathscr{A}_e & \xrightarrow{\alpha_{\bar{e}}} & \mathscr{A}_b \\ \Big\downarrow{\scriptstyle \mathrm{ad}\,(\delta_e^{-1})\circ\phi_a} & & \Big\downarrow{\scriptstyle \varepsilon} & & \Big\downarrow{\scriptstyle \mathrm{ad}\,(\delta_{\bar{e}}^{-1})\circ\phi_b} \\ \mathscr{A}_a & \xleftarrow[\alpha_e]{} & \mathscr{A}_e & \xrightarrow[\alpha_{\bar{e}}]{} & \mathscr{A}_b \end{array} \tag{10}$$

This imposes a non-trivial compatibility on the choices of δ_e and $\delta_{\bar{e}}$. Thus

$$\begin{aligned} \mathrm{Im}(\phi_V) &= \prod_{a\in VA}{}' \mathrm{Aut}^E(\mathscr{A}_e) \\ &:= \left\{ (\phi_a) \in \prod_{a\in VA} \mathrm{Aut}(\mathscr{A}_a) \;\middle|\; \begin{array}{l} \forall e \in EA,\ \partial_0 e = a,\ \partial_1 e = b, \exists \delta_e \in \mathscr{A}_a,\ \delta_{\bar{e}} \in \mathscr{A}_b, \\ \text{and } \varepsilon \in \mathrm{Aut}(\mathscr{A}_e) \text{ such that (10) commutes.} \end{array} \right\} \end{aligned} \tag{11}$$

Similarly, $\mathrm{Im}(\phi_{(V)})$ is the corresponding quotient of (11) mod $\Pi_a \mathrm{ad}(\mathscr{A}_a)$:

$$\begin{aligned} \mathbf{G}^A/\mathbf{G}^{(V)} = \delta\mathbf{G}^A/\delta\mathbf{G}^{(V)} &\cong \prod_{a\in VA}{}' \mathrm{Out}^E(\mathscr{A}_a) \\ &\cong \mathrm{Out}(\Gamma_a)^A_{I_a}/\mathrm{Out}(\Gamma_a)^{(V)}_{I_a} \quad (\forall a \in VA). \end{aligned} \tag{12}$$

7. A filtration structure on $\mathrm{Out}(\Gamma)_I$

In order to introduce a useful filtration between $\mathrm{In}\,\mathbf{G}$ and $\delta\,\mathbf{G}^{(V)}$ we here introduce an auxiliary group Λ, an epimorphism $D : \Lambda \to \delta\mathbf{G}^{(V)}$, and a filtration of Λ. The results of these calculations are summarized in Theorem 8.1 below.

7.1. The group Λ. For $a \in VA$ and $e \in E_0(a)$ we shall use the notation

$$N_e = N_{\mathscr{A}_a}(\alpha_e \mathscr{A}_e) \qquad \text{(normalizer)} \tag{1}$$

$$\begin{aligned} Z_e &= Z_{\mathscr{A}_a}(\alpha_e \mathscr{A}_e) \qquad \text{(centralizer)} \\ Z_{(e)} &= Z(\mathscr{A}_e) \quad \text{and} \quad Z_a = Z(\mathscr{A}_a) \quad \text{(centers).} \end{aligned} \tag{2}$$

We define a homomorphism $\mathrm{ad}_{\mathscr{A}_e} : N_e \to \mathrm{Aut}(\mathscr{A}_e)$, by

$$\alpha_e(\mathrm{ad}_{\mathscr{A}_e}(\sigma)(s)) = \sigma\alpha_e(s)\sigma^{-1}. \tag{3}$$

Now define

$$\Lambda_a = \left(\prod_{e\in E_0(a)} N_e \right) \times \mathscr{A}_a. \tag{4}$$

For $\lambda_a = ((\sigma_e)_{e\in E_0(a)}, h_a) \in \Lambda_a$, define

$$\begin{aligned} &\phi_a(=\phi_a(\lambda_a)) = \mathrm{ad}(h_a) \in \mathrm{Aut}(\mathscr{A}_a), \\ &\phi_e(=\phi_e(\lambda_a)) = \mathrm{ad}_{\mathscr{A}_e}(\sigma_e) \in \mathrm{Aut}(\mathscr{A}_e). \end{aligned} \tag{5}$$

Now define

$$\begin{aligned} \Lambda &= \prod_{a\in VA}{}' \Lambda_a \\ &:= \left\{ (\lambda_a)_{a\in VA} \in \prod_{a\in VA} \Lambda_a \,\middle|\, \forall e \in EA,\ \partial_0 e = a,\ \partial_1 e = b,\ \phi_e(\lambda_a) = \phi_{\bar{e}}(\lambda_b) \right\} \end{aligned} \tag{6}$$

We next define the homomorphism

$$D : \Lambda \to \delta\mathbf{G}^{(V)} \tag{7}$$

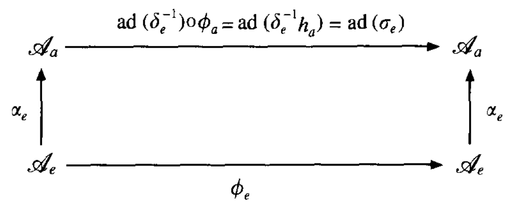
on $\lambda = (\lambda_a)_{a\in VA}$, $\lambda_a = ((\sigma_e)_{e\in E_0(a)}, h_a)$, by

$$\begin{aligned} &D(\lambda) = \Phi^\lambda = (\phi^\lambda, (\delta^\lambda)) \quad \text{where, for } a \in VA,\ e \in E_0(a), \\ &\phi^\lambda_A = \mathrm{Id}_A, \quad \phi^\lambda_a = \phi_a(\lambda_a), \quad \phi^\lambda_e = \phi_e(\lambda_a), \\ &\delta^\lambda_a = 1 \quad \text{and} \quad \delta^\lambda_e = h_a\sigma_e^{-1}. \end{aligned} \tag{8}$$

The conditions of (6) and (8) and 6.1(4) show that in fact, $\Phi^\lambda \in \delta\mathbf{G}^{(V)}$. It is easily seen that D is a homomorphism. We next show that

$$D : \Lambda \to \delta\mathbf{G}^{(V)} \quad \text{is surjective.} \tag{9}$$

In fact, let $\Phi = (\phi, (\delta)) \in \delta\mathbf{G}^{(V)}$. By definition of $\mathbf{G}^{(V)}$ (6.1(4)), $\phi_a = \mathrm{ad}(h_a)$ for some $h_a \in \mathscr{A}_a$. For $e \in E_0(a)$ put $\sigma_e = \delta_e^{-1}h_a$. The commutativity of the diagram



shows then that $\sigma_e \in N_e$ and $\phi_e = \mathrm{ad}_{\mathscr{A}_e}(\sigma_e)$. Since $\phi_e = \phi_{\bar{e}}$ it follows that $\lambda = (\lambda_a)_{a\in VA}$ defined by $\lambda_a = ((\sigma_e)_{e\in E_0(a)}, h_a)$ defines an element $\lambda \in \Lambda$ such that $D(\lambda) = \Phi$.

Finally, we calculate Ker(D). Since $Z_a := Z(\mathscr{A}_a) \le N_e\ \forall e \in E_0(a)$, we have the diagonal homomorphism

$$\Delta_a : Z_a \to \Lambda_a, \quad \Delta_a(z) = ((\sigma_e)_e, h_a) \quad \text{with} \quad h_a = z = \sigma_e\ \forall e \in E_0(a). \tag{10}$$

Since, evidently, $\phi_a(\Delta_a(z)) = \mathrm{Id}_{\mathscr{A}_a}$, $\phi_e(\Delta_a(z)) = \mathrm{Id}_{\mathscr{A}_e}$, we have

$$\Delta Z_V := \prod_{a\in VA} \Delta_a Z_a \le \Lambda. \tag{11}$$

From (8) we see that $\Delta Z_V \leq \mathrm{Ker}(D)$. We claim that this is an equality. For suppose $\lambda = (\lambda_a)$ as above and $D(\lambda) = \Phi^\lambda = I$. Then from (8) we see that $\mathrm{Id}_{\mathscr{A}_a} = \phi_a = \mathrm{ad}(h_a)$, so $h_a \in Z_a$, and $1 = \delta_e = h_a \sigma_e^{-1}$, so $\sigma_e = h_a$ for $e \in E_0(a)$. Thus $\lambda_a = \Delta_a(h_a)$, so $\lambda \in \Delta Z_V$.

In summary, putting

$$Z_V = \prod_{a \in VA} Z_a \tag{12}$$

and defining $\Delta = (\Delta_a) : Z_V \to \prod_{a \in VA} \Lambda_a$, we have an exact sequence

$$1 \to Z_V \xrightarrow{\Delta} \Lambda \xrightarrow{D} \delta\mathbf{G}^{(V)} \to 1. \tag{13}$$

7.2. The quotient $\Lambda/\Lambda^{(E)}$. Let

$$\begin{aligned} \lambda &= (\lambda_a)_{a \in V\Lambda} \in \Lambda, \\ \lambda_a &= ((\sigma_e)_{e \in E_0(a)}, h_a) \in \Lambda_a = \left(\prod_{e \in E_0(a)} N_e \right) \times \mathscr{A}_a. \end{aligned} \tag{1}$$

Recall that

$$\phi_e = \phi_e(\lambda) := \mathrm{ad}_{\mathscr{A}_e}(\sigma_e) \in \mathrm{Aut}(\mathscr{A}_e). \tag{2}$$

Clearly

$$\phi_e(\lambda) \in \mathrm{ad}(\mathscr{A}_e) \Longleftrightarrow \sigma_e \in (\alpha_e \mathscr{A}_e) \cdot Z_e, \tag{3}$$

where $Z_e = Z_{\mathscr{A}_a}(\alpha_e \mathscr{A}_e)$.

We have a homomorphism

$$\begin{aligned} \Lambda &\to \prod_{e \in EA} \mathrm{ad}_{\mathscr{A}_e}(N_e) \\ \lambda &\longmapsto (\phi_e(\lambda))_{e \in EA} \end{aligned} \tag{4}$$

whose image is

$$\prod_{e \in EA}{}' \mathrm{ad}_{\mathscr{A}_e}(N_e) := \left\{ (\phi_e) \in \prod_e \mathrm{ad}_{\mathscr{A}_e}(N_e) \,\middle|\, \phi_e = \phi_{\bar{e}} \quad \forall e \in EA \right\}. \tag{5}$$

The inverse image of the inner automorphisms is

$$\Lambda^{(E)} := \{ \lambda \in \Lambda \mid \phi_e(\lambda) \in \mathrm{ad}(\mathscr{A}_e) \ \forall e \}. \tag{6}$$

Thus we have an isomorphism

$$\Lambda/\Lambda^{(E)} \xrightarrow{\cong} \prod_{e \in EA}{}' (\mathrm{ad}_{\mathscr{A}_e}(N_e)/\mathrm{ad}(\mathscr{A}_e)) := \left(\prod_e{}' \mathrm{ad}_{\mathscr{A}_e}(N_e) \right) \Big/ \left(\prod_e{}' \mathrm{ad}(\mathscr{A}_e) \right). \tag{7}$$

Defining the *geometric edges of A* by

$$GEA := \{\{e, \bar{e}\} \mid e \in EA\},$$

we obtain from (7) and the definition (5) of $\prod'$ an isomorphism

$$\Lambda/\Lambda^{(E)} \xrightarrow{\cong} \prod_{\{e,\bar{e}\}\in GEA} \frac{\mathrm{ad}_{\mathscr{A}_e}(N_e) \cap \mathrm{ad}_{\mathscr{A}_e}(N_{\bar{e}})}{\mathrm{ad}(\mathscr{A}_e)}. \tag{8}$$

Next observe that, for the homomorphism

$$D : \Lambda \to \delta\mathbf{G}^{(V)}$$

we have

$$\Lambda^{(E)} = D^{-1}(\delta\mathbf{G}^{(V,E)}) \tag{9}$$

(cf. 6.1(5)). Hence we have isomorphisms

$$\Lambda/\Lambda^{(E)} \cong \delta\mathbf{G}^{(V)}/\delta\mathbf{G}^{(V,E)} \cong \mathrm{Out}(\Gamma_a)_{l_a}^{(V)}/\mathrm{Out}(\Gamma_a)_{l_a}^{(V,E)}. \tag{10}$$

7.3. The group $\Lambda^{[E]} \leq \Lambda^{(E)}$ is defined by

$$\Lambda^{[E]} = \{\lambda \in \Lambda \mid \sigma_e \in \alpha_e \mathscr{A}_e \ \forall e \in EA\}. \tag{1}$$

For $\lambda \in \Lambda^{[E]}$, λ as in 7.2(1), put

$$\sigma_e = \alpha_e(s_e), \quad s_e \in \mathscr{A}_e. \tag{2}$$

Then

$$\phi_e = \mathrm{ad}(s_e), \quad \text{and so, since } \phi_e = \phi_{\bar{e}}, \quad z_e := s_e^{-1}s_{\bar{e}} \in Z_{(e)} := Z(\mathscr{A}_e) = z_{\bar{e}}^{-1}. \tag{3}$$

For $D(\lambda) = \Phi^\lambda = (\phi, (\delta))$, we have $\delta_e = h_a\alpha_e(s_e)^{-1}$. It follows from Theorem 4.1 that

$$\delta\mathbf{G}^{(V,E]} := D(\Lambda^{[E]}) \geq \delta\,\mathrm{In}\,\mathbf{G}. \tag{4}$$

Hence, putting

$$\mathrm{Out}(\Gamma_a)_{l_a}^{(V,E]} := \sigma_a(\delta\mathbf{G}^{(V,E]}), \tag{5}$$

we have

$$\delta\mathbf{G}^{(V,E)}/\delta\mathbf{G}^{(V,E]} \cong \mathrm{Out}(\Gamma_a)_{l_a}^{(V,E)}/\mathrm{Out}(\Gamma_a)_{l_a}^{(V,E]}. \tag{6}$$

Now $\Lambda^{(E)}/\Lambda^{[E]}$ maps onto $\delta\mathbf{G}^{(V,E)}/\delta\mathbf{G}^{(V,E]}$, but this may not be injective, since $\Lambda^{[E]}$ need not contain $\mathrm{Ker}(D) = \Delta Z_V$. Instead we have

$$\delta\mathbf{G}^{(V,E)}/\delta\mathbf{G}^{(V,E]} \cong \Lambda^{(E)}/\Lambda^{[E]} \cdot \Delta Z_V. \tag{7}$$

We now analyze the right-hand side of (7). First note that

$$\Lambda^{(E)} = \prod_{a\in VA}{}' \Lambda_a^{(E)}, \quad \text{where } \Lambda_a^{(E)} = \left(\prod_{e\in E_0(a)} (\alpha_e\mathscr{A}_e)\cdot Z_e\right) \times \mathscr{A}_a, \qquad Z_e = Z_{\mathscr{A}_a}(\alpha_e\mathscr{A}_e), \tag{8}$$

$$\Lambda^{[E]} = \prod_{a\in VA}{}' \Lambda_a^{[E]}, \quad \text{where } \Lambda_a^{[E]} = \left(\prod_{e\in E_0(a)} \alpha_e\mathscr{A}_e\right) \times \mathscr{A}_a. \tag{9}$$

The $\prod'$ notation designates the restriction needed to make $\phi_e = \phi_{\bar{e}}$. Since $Z_e \cap \alpha_e\mathscr{A}_e = \alpha_e Z_{(e)}$, $Z_{(e)} = Z(\mathscr{A}_e)$, we have $(\alpha_e\mathscr{A}_e)\cdot Z_e/\alpha_e\mathscr{A}_e \cong Z_e/\alpha_e Z_{(e)}$, and so

$$\Lambda^{(E)}/\Lambda^{[E]} \cong \prod_a \prod_{e\in E_0(a)} Z_e/\alpha_e Z_{(e)}. \tag{10}$$

Here the $'$ has been omitted on the first product, since the factors from Z_e will never affect the compatibility conditions, $\phi_e = \phi_{\bar{e}}$.

Next observe that

$$\Lambda_a^{[E]}\cdot(\Delta_a Z_a) = \left[\left(\prod_{e\in E_0(a)} \alpha_e\mathscr{A}_e\right)\cdot(\Delta_{E_0(a)}Z_a)\right] \times \mathscr{A}_a,$$

where

$$\Delta_{E_0(a)}Z_a = \operatorname{Im}\left(\Delta : Z_a \to \prod_{e\in E_0(a)} Z_e\right). \tag{11}$$

From (10) and (11), and (6) and (7), we conclude that

$$\begin{aligned}\Lambda^{(E)}/\Lambda^{[E]}\cdot\Delta Z_V &\cong \prod_a \frac{\prod_{e\in E_0(a)} Z_e}{\left(\prod_{e\in E_0(a)}\alpha_e Z_{(e)}\right)\cdot\Delta_{E_0(a)}Z_a}\\ &\cong \delta\mathbf{G}^{(V,E)}/\delta\mathbf{G}^{(V,E]}\\ &\cong \operatorname{Out}(\Gamma_a)_{l_a}^{(V,E)}/\operatorname{Out}(\Gamma_a)_{l_a}^{(V,E]}.\end{aligned} \tag{12}$$

7.4. The group $\Lambda^{[EZ]}$. For $\lambda \in \Lambda^{[E]}$ as in 7.3, we have from 7.3(3)

$$z_e(= z_e(\lambda)) = s_e^{-1}s_{\bar{e}} \in Z_{(e)} = Z(\mathscr{A}_e). \tag{1}$$

Suppose that $\lambda' \in \Lambda^{[E]}$ and $\lambda'' = \lambda'\lambda$. Then $z_e(\lambda'') = (s_e'')^{-1}s_{\bar{e}}'' = (s_e's_e)^{-1}(s_{\bar{e}}'s_{\bar{e}}) = s_e^{-1}s_e'^{-1}s_{\bar{e}}'s_{\bar{e}} = s_e^{-1}z_e(\lambda')s_{\bar{e}} = z_e(\lambda')s_e^{-1}s_{\bar{e}} = z_e(\lambda')z_e(\lambda)$. Thus we have a *homomorphism*

$$\begin{aligned}\zeta : \Lambda^{[E]} &\to \prod_{e\in EA}{}' Z_{(e)}\\ \lambda &\longmapsto (z_e(\lambda))_{e\in EA}.\end{aligned} \tag{2}$$

Here the $\prod'$ notation designates the restriction that $z_{\bar{e}} = z_e^{-1} \quad \forall e \in EA$ (cf. 7.3(3)). If, in λ, we replace each s_e by $s'_e = s_e w_e$, with $w_e \in Z_{(e)}$, we obtain a new element $\lambda' \in \Lambda^{[E]}$ with $z_e(\lambda') = z_e(\lambda) \cdot (w_e^{-1} w_{\bar{e}})$. Since we can freely choose the $w'_e s$, it follows that

homomorphism ζ is surjective. (3)

Now define

$$\Lambda^{[EZ]} = \{\lambda \in \Lambda^{[E]} \mid z_e(\lambda) \in Z_e(\mathfrak{A}) \quad \forall e \in EA\}. \tag{4}$$

Recall from 6.0(11) that $Z_e(\mathfrak{A})$ is defined by

$$\alpha_e Z_e(\mathfrak{A}) = Z_a(\mathfrak{A}) := Z(\Gamma_a), \tag{5}$$

for $a = \partial_0 e$. We put

$$\begin{aligned} \delta\mathbf{G}^{(V,EZ]} &= D(\Lambda^{[EZ]}), \\ \mathrm{Out}(\Gamma_a)_{l_a}^{(V,EZ]} &= \sigma_a(\delta\mathbf{G}^{(V,EZ]}). \end{aligned} \tag{6}$$

It follows from Theorem 6.4 that

$$\delta \mathrm{In}\, \mathbf{G} \le \delta\mathbf{G}^{(V,EZ]}, \tag{7}$$

and so

$$\delta\mathbf{G}^{(V,E]}/\delta\mathbf{G}^{(V,EZ]} \cong \mathrm{Out}(\Gamma_a)_{l_a}^{(V,E]}/\mathrm{Out}(\Gamma_a)_{l_a}^{(V,EZ]}. \tag{8}$$

From (6) and 7.3(4) we see that the groups in (8) are a quotient of $\Lambda^{[E]}/\Lambda^{[EZ]}$. In view of (3) and the definition (4) of $\Lambda^{[EZ]}$ as $\zeta^{-1}(\prod'_e Z_e(\mathfrak{A}))$, we have a ζ-induced isomorphism

$$\bar{\zeta} : \Lambda^{[E]}/\Lambda^{[EZ]} \xrightarrow{\cong} \prod_{e \in EA}{}' Z_{(e)}/Z_e(\mathfrak{A}). \tag{9}$$

From 7.1(13) we have

$$\mathrm{Ker}(\Lambda^{[E]} \xrightarrow{D} \delta\mathbf{G}^{(V,E]}) = \Lambda^{[E]} \cap \Delta Z_V. \tag{10}$$

It is clear from the definitions 7.3(1) and 7.1(10) and (11) of the latter two groups that

$$\Lambda^{[E]} \cap \Delta Z_V = \prod_{a \in VA} \Delta_a Z_{aE}, \quad \text{where } Z_{aE} := Z_a \cap \bigcap_{e \in E_0(a)} \alpha_e \mathscr{A}_e. \tag{11}$$

Thus, putting $Z_{VE} = \prod_{a \in VA} Z_{aE}$, we have

$$\mathrm{Ker}(\Lambda^{[E]} \xrightarrow{D} \delta\mathbf{G}^{(V,E]}) = \text{“}\Delta Z_{VE}\text{”} := \prod_{a \in VA} \Delta_a Z_{aE}, \tag{12}$$

and D induces an isomorphism

$$\Lambda^{[E]}/\Lambda^{[EZ]} \cdot \Delta Z_{VE} \xrightarrow{\cong} \delta\mathbf{G}^{(V,E]}/\delta\mathbf{G}^{(V,EZ]}. \tag{13}$$

Combining (9) and (13) we see that

$$\delta\mathbf{G}^{(V,E]}/\delta\mathbf{G}^{(V,EZ]} \cong \operatorname{Coker}\left(Z_{VE} = \prod_{a\in VA} Z_{aE} \xrightarrow{\omega} \prod_{e\in EA}{}' \frac{Z_{(e)}}{Z_e(\mathfrak{A})} \right) \tag{14}$$

where, for $w = (w_a)_{a\in VA}$, $w_a \in Z_{aE}$, and $w_a = \alpha_e(w_e)$ for $e \in E_0(a)$, we put $\tilde{\omega}_e(w) = w_e^{-1}w_{\bar{e}} \in Z_{(e)}$, and define $\omega(w) = (\omega_e(w))_{e\in EA}$, where $\omega_e(w)$ denotes the class of $\tilde{\omega}_e(w) \bmod Z_e(\mathfrak{A})$.

7.5. The homomorphism $\Lambda^{[EZ]} \to \operatorname{Hom}(\pi_1(A,a), Z(\mathfrak{A}))$. Recall the surjection induced by 7.4(2) and (4),

$$\zeta : \Lambda^{[EZ]} \to \prod_{e\in EA}{}' Z_e(\mathfrak{A}), \tag{1}$$

where

$$\prod_{e\in EA}{}' Z_e(\mathfrak{A}) = \left\{ z = (z_e) \in \prod_{e\in EA} Z_e(\mathfrak{A}) \,\middle|\, z_{\bar{e}} = z_e^{-1} \ \forall e \in EA \right\}. \tag{2}$$

For $z = (z_e)_{e\in EA}$ as in (2), each z_e defines an element $z(e) \in Z(\mathfrak{A})$ (cf. 6.0(12)) with components $z_u(e) \in Z_u(\mathfrak{A})$ $\forall u \in VA \cup EA$, and $z_e(e) = z_e$.

Recall the path group of the graph A (a graph of trivial groups),

$$\pi(A) = \langle EA \mid e\bar{e} = 1 \ \forall e \in EA \rangle. \tag{3}$$

It follows that an element $z \in \prod'_{e\in EA} Z_e(\mathfrak{A})$ defines (in fact, is equivalent to) a homomorphism

$$\chi_z : \pi(A) \to Z(\mathfrak{A}), \qquad \chi_z(e) = z(e). \tag{4}$$

Moreover, $z \mapsto \chi_z$ defines a homomorphism, in fact, an isomorphism,

$$\chi : \prod_{e\in EA}{}' Z_e(\mathfrak{A}) \xrightarrow{\cong} \operatorname{Hom}(\pi(A), Z(\mathfrak{A})). \tag{5}$$

Let $a \in VA$, so that $\pi_1(A,a) \le \pi(A)$. Then we have the composite homomorphism

$$\begin{array}{ccc} \prod'_{e\in EA} Z_e(\mathfrak{A}) & \xrightarrow[\cong]{\chi} & \operatorname{Hom}(\pi(A), Z(\mathfrak{A})) \\ \uparrow \zeta & & \downarrow \operatorname{res}_a \\ \Lambda^{[EZ]} & \xrightarrow[\mu_a]{} & \operatorname{Hom}(\pi_1(A,a), Z(\mathfrak{A})) \end{array} \tag{6}$$

It is easily seen that res_a is surjective, hence

$$\mu_a \text{ is surjective.} \tag{7}$$

If $g \in \pi[b,a]$ then $\mathrm{ad}(g) : \pi_1(A,a) \to \pi_1(A,b)$, and $\mu_a = \mu_b \circ \mathrm{ad}(g)$. Hence $\mathrm{Ker}(\mu_a)$ is independent of a. We put

$$\mathrm{In}\,\Lambda = \mathrm{Ker}(\mu_a : \Lambda^{[EZ]} \to \mathrm{Hom}(\pi_1(A,a), Z(\mathfrak{A}))). \tag{8}$$

The point of this notation is that it follows from Theorem 6.4 that

$$D(\mathrm{In}\,\Lambda) = \delta\,\mathrm{In}\,\mathbf{G}. \tag{9}$$

Claim. $\mathrm{Ker}(\Lambda^{[EZ]} \xrightarrow{D} \delta\mathbf{G}^{(V,EZ]}) \leq \mathrm{In}\,\Lambda.$ (10)

Say $\lambda \in \Lambda^{[EZ]} \cap \mathrm{Ker}(D) = \Lambda^{[EZ]} \cap \Delta Z_V$. Then $\lambda = (\lambda_a)_{a\in VA}$ where $\lambda_a = \Delta_a h_a$ with $h_a \in Z_{aE} = Z_a \cap \bigcap_{e\in E_0(a)} \alpha_e \mathscr{A}_e$, say $h_a = \alpha_e(h_e)$, $h_e \in \mathscr{A}_e$, and we have

$$z_e := h_e^{-1} h_{\bar{e}} \in Z_e(\mathfrak{A}) \quad \forall e \in EA. \tag{11}$$

Each $h_a \in Z_a(\mathfrak{A}) = Z(\Gamma_a)$ defines an element $z(a) \in Z(\mathfrak{A})$ with $z_a(a) = h_a$ and $z_e(a) = h_e$ for $e \in E_0(a)$. The element $z = (z_e)_{e\in EA}$ defines (cf. (4)) $\chi_z : \pi(A) \to Z(\mathfrak{A})$ by $\chi_z(e)_e = h_e^{-1} h_{\bar{e}} = z_e(a)^{-1} z_{\bar{e}}(b)$ $(b = \partial_1 e)$, whence

$$\chi_z(e) = z(a)^{-1} z(b) \quad (a = \partial_0 e,\ b = \partial_1 e). \tag{12}$$

It follows then that,

If $\gamma = (e_1, e_2, \ldots, e_n)$ is a path in A from a to b then $\chi_z(|\gamma|) = z(a)^{-1} z(b)$. (13)

Hence, $\chi_z(|\gamma|) = 1$ if γ is a closed path $(a = b)$, and so

$$\chi_z|_{\pi_1(A,a)} \text{ is trivial,} \tag{14}$$

whence (10) (cf. definition (8)).

Finally, combining (7)–(10) and 7.4(6), we obtain isomorphisms

$$\begin{aligned} \mathrm{Out}(\Gamma_a)_{l_a}^{(V,EZ]} &\cong \delta\mathbf{G}^{(V,EZ]}/\delta\,\mathrm{In}\,\mathbf{G} \cong \Lambda^{[EZ]}/\mathrm{In}\,\Lambda \\ &\cong \mu_a(\Lambda^{[EZ]}) = \mathrm{Hom}(\pi_1(A,a), Z(\mathfrak{A})). \end{aligned} \tag{15}$$

8. Filtration summary

The next theorem assembles all of our calculations of the successive quotients for the $\mathrm{Out}(\Gamma_a)_{l_a}$-filtration.

8.1. Theorem. *Let $\mathfrak{A} = (A, \mathscr{A})$ be a minimal non-abelian graph of groups, $a \in VA$, $\Gamma_a = \pi_1(\mathfrak{A}, a)$, $X_a = (\widetilde{\mathfrak{A}, a})$, and let l_a denote the hyperbolic length function of the Γ_a-action on X_a. Let*

(1) $H = \mathrm{Out}(\Gamma_a)_{l_a} =$ *the stabilizer of l_a in* $\mathrm{Out}(\Gamma_a) = \mathrm{Aut}(\Gamma_a)/\mathrm{ad}(\Gamma_a)$.

There is a filtration,

(2) $H \triangleright H^A \triangleright H^{(V)} \triangleright H^{(V,E)} \triangleright H^{(V,E]} \triangleright H^{(V,EZ]} \triangleright \{1\}$, *with successive quotients described as follows:*

(3) $H/H^A \leq \mathrm{Aut}(A)$ (6.6(5))

(4) $H^A/H^{(V)} \cong \prod'_{a \in VA} \mathrm{Out}^E(\mathscr{A}_a)$ (6.7(12))

(5) $H^{(V)}/H^{(V,E)} \cong \prod_{\{e,\bar{e}\} \in GEA} \dfrac{\mathrm{ad}_{\mathscr{A}_e}(N_e) \cap \mathrm{ad}_{\mathscr{A}_e}(N_{\bar{e}})}{\mathrm{ad}(\mathscr{A}_e)}$ (7.1(8) *and* (10))

(6) $H^{(V,E)}/H^{(V,E]} \cong \prod_{a \in VA} \dfrac{\prod_{e \in E_0(a)} Z_e}{\left(\prod_{e \in E_0(a)} \alpha_e Z_{(e)}\right) \cdot (\Delta_{E_0(a)} Z_a)}$ (7.3(12))

(7) $H^{(V,E]}/H^{(V,EZ]} \cong \mathrm{Coker}\left(Z_{VE} = \prod_{a \in VA} Z_{aE} \xrightarrow{\omega} \prod'_{e \in EA} \frac{Z_{(e)}}{Z_e(\mathfrak{A})}\right)$ (7.4(14))

(8) $H^{(V,EZ]} \cong \mathrm{Hom}(\pi_1(A,a), Z(\mathfrak{A}))$ (7.5(15))

8.2. Explanation of notation. We collect here, in one place, the definitions of the groups occurring in Theorem 8.1.

In Theorem 8.1(3), $\mathrm{Aut}(A)$ denotes the group of automorphisms of the graph A. When A is finite, e.g. when Γ_a is finitely generated, $\mathrm{Aut}(A)$ is finite.

In Theorem 8.1(4), $\mathrm{Out}^E(\mathscr{A}_a) = \mathrm{Aut}^E(\mathscr{A}_a)/\mathrm{ad}(\mathscr{A}_a)$, where

$$\mathrm{Aut}^E(\mathscr{A}_a) = \{\phi \in \mathrm{Aut}(\mathscr{A}_a) \mid \phi\alpha_e\mathscr{A}_e \text{ is } \mathscr{A}_a\text{-conjugate to } \alpha_e\mathscr{A}_e \ \forall e \in E_0(a)\}.$$

Then

$$\prod_{a \in VA}{}' \mathrm{Out}^E(\mathscr{A}_a) = \left(\prod_{a \in VA}{}' \mathrm{Aut}^E(\mathscr{A}_a)\right) \Big/ \left(\prod_{a \in VA} \mathrm{ad}(\mathscr{A}_a)\right),$$

where $(\phi_a)_{a \in VA} \in \prod_{a \in VA} \mathrm{Aut}(\mathscr{A}_a)$ belongs to $\prod'_{a \in VA} \mathrm{Aut}^E(\mathscr{A}_a)$ iff, $\forall e \in EA$, $\partial_0 e = a$, $\partial_1 e = b$, $\exists \delta_e \in \mathscr{A}_a$, $\delta_{\bar{e}} \in \mathscr{A}_b$, and $\varepsilon \in \mathrm{Aut}(\mathscr{A}_e)$ such that the following diagram commutes:

$$\begin{array}{ccccc}
\mathscr{A}_a & \xleftarrow{\alpha_e} & \mathscr{A}_e & \xrightarrow{\alpha_{\bar{e}}} & \mathscr{A}_b \\
\Big\downarrow {\scriptstyle \mathrm{ad}(\delta_e^{-1}) \circ \phi_a} & & \Big\downarrow {\scriptstyle \varepsilon} & & \Big\downarrow {\scriptstyle \mathrm{ad}(\delta_{\bar{e}}^{-1}) \circ \phi_b} \\
\mathscr{A}_a & \xleftarrow[\alpha_e]{} & \mathscr{A}_e & \xrightarrow[\alpha_{\bar{e}}]{} & \mathscr{A}_b
\end{array}$$

In Theorem 8.1(5), $N_e = N_{\mathscr{A}_a}(\alpha_e\mathscr{A}_e)$, and $\mathrm{ad}_{\mathscr{A}_e} : N_e \to \mathrm{Aut}(\mathscr{A}_e)$ is defined by $\alpha_e(\mathrm{ad}_{\mathscr{A}_e}(\sigma)(s)) = \sigma\alpha_e(s)\sigma^{-1}$, for $\sigma \in N_e$, $s \in \mathscr{A}_e$. We similarly define $\mathrm{ad}_{\mathscr{A}_e} : N_{\bar{e}} \to$

$\mathrm{Aut}(\mathscr{A}_e)$. The notation GEA designates the geometric edges of A : $GEA = \{\{e, \bar{e}\} \mid e \in EA\}$.

In Theorem 8.1(6), $Z_e = Z_{\mathscr{A}_a}(\alpha_e \mathscr{A}_e)$ $(a = \partial_0 e)$, $Z_{(e)} = Z(\mathscr{A}_e)$, $Z_a = Z(\mathscr{A}_a)$, and $\Delta_{E_0(a)} : Z_a \to \prod_{e \in E_0(a)} Z_e$ is the diagonal embedding.

In Theorem 8.1(7), $Z_e(\mathfrak{A})$ is defined, as in 6.0(11), by $\alpha_e Z_e(\mathfrak{A}) = Z(\Gamma_a) =: Z_a(\mathfrak{A})(a = \partial_0 e)$. Further $Z_{aE} = Z_a \cap \bigcap_{e \in E_0(a)} \alpha_e \mathscr{A}_e$. For $z_a \in Z_{aE}$ we have $z_a = \alpha_e z_e$ with $z_e \in Z_{(e)}$, and so we can define a homomorphism

$$\tilde{\omega} : Z_{VE} := \prod_{a \in VA} Z_{aE} \to \prod_{e \in EA}{}' Z_{(e)},$$

$$\tilde{\omega}((z_a)_{a \in VA}) = ((z_e^{-1} z_{\bar{e}})_{e \in EA}),$$

and $\prod'_{e \in EA} Z_{(e)}$ consists of all $(w_e)_{e \in EA} \in \prod_{e \in EA} Z_{(e)}$ such that $w_{\bar{e}} = w_e^{-1}$ $\forall e \in EA$. We have

$$\prod_{e \in EA}{}' Z_{(e)}/Z_e(\mathfrak{A}) := \left(\prod_{e \in EA}{}' Z_{(e)} \right) \Big/ \left(\prod_{e \in EA}{}' Z_e(\mathfrak{A}) \right),$$

and $\omega : Z_{VE} \to \prod_{e \in EA} Z_{(e)}/Z_e(\mathfrak{A})$ is obtained, by passage to the quotient, from $\tilde{\omega}$.

In Theorem 8.1(8), $Z(\mathfrak{A})$ is defined as in 6.0(10).

Some of the groups above are nested as follows, for $a \in VA$, $e \in E_0(a)$,

$$\Gamma_a \geq \mathscr{A}_a \geq N_e \rhd Z_e \rhd \alpha_e Z_{(e)} \rhd Z_a(\mathfrak{A}) \rhd \{1\},$$

$$Z_e \rhd Z_a \quad \rhd Z_a(\mathfrak{A}).$$

8.3. Remark

(1) In case $Z(\Gamma_a) = \{1\}$, as happens, for example, when Γ_a acts faithfully on X_a, since $Z(\Gamma_a)$ acts trivially on X_a (1.5), we have $Z(\mathfrak{A}) = \{1\}$, so $H^{(V,EZ]} = \{1\}$ in Theorem 8.1(8), and, since $Z_e(\mathfrak{A}) = \{1\}$, we have, from Theorem 8.1(7), an isomorphism

$$H^{(V,E]} \cong \mathrm{Coker}\left(Z_{VE} \to \prod_{e \in EA}{}' Z_{(e)} \right).$$

(2) If A is a tree, so that $\pi_1(A, a) = \{1\}$, then again we have $H^{(V,EZ]} = \{1\}$ in Theorem 8.1(8).

(3) Suppose that all the vertex groups $\mathscr{A}_a$ have trivial centers, $Z_a(= Z(\mathscr{A}_a)) = \{1\}$. Then $Z(\mathfrak{A}) = \{1\}$ also, as in Remark (1) above, so $H^{(V,EZ]} = \{1\}$. Further, Theorem 8.1(6) and (7) simplify as follows:

$$H^{(V,E)}/H^{(V,E]} \cong \prod_{e \in EA} Z_e/\alpha_e Z_{(e)},$$

$$H^{(V,E]} \cong \prod_{e \in EA}{}' Z_{(e)}/Z_e(\mathfrak{A}).$$

8.4. The case of an amalgam (cf. 5.1). Suppose that

$$A = a \circ\!\!\overset{e}{\longrightarrow}\!\!\circ\, b. \tag{1}$$

We shall view α_a and α_b as inclusions of a proper subgroup,

$$\mathscr{A}_a \supsetneq \mathscr{A}_e \subsetneq \mathscr{A}_b \tag{2}$$

and put

$$\Gamma = \mathscr{A}_a *_{\mathscr{A}_e} \mathscr{A}_b = \pi_1(\mathfrak{A}, A) \qquad (2.2(11)). \tag{3}$$

Let l denote the length function of the Γ-action on $X_a = (\widetilde{\mathfrak{A}, a})$, and put

$$H = \mathrm{Out}(\Gamma)_l, \tag{4}$$

which we filter as in Theorem 8.1. We shall make more explicit what Theorem 8.1 tells us in this case.

We have

$$\mathrm{Aut}(A) = \{I, \sigma\}, \quad \text{where } \sigma(e) = \bar{e}. \tag{5}$$

Moreover, it is easily seen that,

$$\begin{array}{l} H/H^A \le \mathrm{Aut}(A), \text{with equality iff there is an isomorphism} \\ \phi : \mathscr{A}_a \to \mathscr{A}_b \ \text{ such that } \ \phi(\mathscr{A}_e) = \mathscr{A}_e. \end{array} \tag{6}$$

For $\phi \in \mathrm{Aut}(\mathscr{A}_a)$, let $[\phi]$ denote its class in $\mathrm{Out}(\mathscr{A}_a) = \mathrm{Aut}(\mathscr{A}_a)/\mathrm{ad}(\mathscr{A}_a)$. Then

$$H^A/H^{(V)} \cong \left\{ (x_a, x_b) \in \mathrm{Out}(\mathscr{A}_a) \times \mathrm{Out}(\mathscr{A}_b) \;\middle|\; \begin{array}{l} \exists \phi_c \in \mathrm{Aut}(\mathscr{A}_c)(c = a, b) \\ \text{such that } \ x_c = [\phi_c] \text{ and} \\ \phi_a|_{\mathscr{A}_e} = \phi_b|_{\mathscr{A}_e} \in \mathrm{Aut}(\mathscr{A}_e) \end{array} \right\}. \tag{7}$$

$$H^{(V)}/H^{(V,E)} \cong \frac{\mathrm{ad}_{\mathscr{A}_e}(N_e) \cap \mathrm{ad}_{\mathscr{A}_e}(N_{\bar{e}})}{\mathrm{ad}(\mathscr{A}_e)}. \tag{8}$$

$$H^{(V,E)}/H^{(V,E]} \cong \left(\frac{Z_e}{\alpha_e Z_{(e)} \cdot Z_a} \right) \times \left(\frac{Z_{\bar{e}}}{\alpha_{\bar{e}} Z_{(e)} \cdot Z_b} \right). \tag{9}$$

In Theorem 8.1(7), $Z_{aE} = Z_a \cap \mathscr{A}_e = Z(\mathscr{A}_a) \cap \mathscr{A}_e := Z_{\mathscr{A}_e}(\mathscr{A}_a)$; similarly $Z_{bE} = Z_{\mathscr{A}_e}(\mathscr{A}_b)$. Evidently

$$Z_{\mathscr{A}_e}(\mathscr{A}_a) \cap Z_{\mathscr{A}_e}(\mathscr{A}_b) = Z(\Gamma) = Z_e(\mathfrak{A}). \tag{10}$$

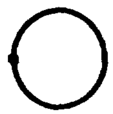
For $w = (w_a, w_b) \in Z_{aE} \times Z_{bE}$ put $\omega_e(w) = w_a^{-1} w_b = \omega_{\bar{e}}(w)^{-1} \in Z_{(e)} = Z(\mathscr{A}_e)$. Then $\omega : Z_{aE} \times Z_{bE} \to (Z_{(e)}/Z_e(\mathfrak{A})) \times' (Z_{(\bar{e})}/Z_e(\mathfrak{A}))$ is induced by $\tilde{\omega} : Z_{aE} \times Z_{bE} \to Z_{(e)} \times' Z_{(\bar{e})}$, $\tilde{\omega}(w) = (\omega_e(w), \omega_{\bar{e}}(w))$. Since the first coordinate in $Z_{(e)} \times' Z_{(\bar{e})}$ determines the second, and $\omega_e(Z_{aE} \times Z_{bE}) = Z_{\mathscr{A}_e}(\mathscr{A}_a) \cdot Z_{\mathscr{A}_e}(\mathscr{A}_b)$ contains $Z_e(\mathfrak{A})$, it follows from Theorem 8.1 (7) that

$$H^{(V,E]}/H^{(V,EZ]} \cong \frac{Z_{(e)}}{Z_{aE} \cdot Z_{bE}} = \frac{Z(\mathscr{A}_e)}{Z_{\mathscr{A}_e}(\mathscr{A}_a) \cdot Z_{\mathscr{A}_e}(\mathscr{A}_b)}. \quad (11)$$

Finally, since A is a tree (cf. Remark 8.3(2)),

$$H^{(V,EZ]} = \{1\}. \quad (12)$$

8.5. The case of an HNN-extension (cf. 5.5). Let

$A = a$  e.

$$\mathscr{A}_e \underset{\alpha_{\bar{e}}}{\overset{\alpha_e}{\rightrightarrows}} \mathscr{A}_a \quad (1)$$

$$\Gamma = \pi_1(\mathfrak{A}, a) = \langle \mathscr{A}_a, e \mid e\alpha_{\bar{e}}(s)e^{-1} = \alpha_e(s) \quad \forall s \in \mathscr{A}_e \rangle. \quad (2)$$

Let l denote the length function of the Γ-action on $X = (\widetilde{\mathfrak{A}, a})$, and

$$H = \mathrm{Out}(\Gamma)_l, \quad (3)$$

which we filter as in Theorem 8.1. We have

$$\mathrm{Aut}(A) = \{I, \sigma\}, \quad \sigma(e) = \bar{e}, \quad (4)$$

and

$$\begin{array}{l} H/H^A \le \mathrm{Aut}(A), \text{ with equality iff} \\ \exists \phi \in \mathrm{Aut}(\mathscr{A}_a) \text{ such that } \phi(\alpha_e \mathscr{A}_e) = \alpha_{\bar{e}} \mathscr{A}_e. \end{array} \quad (5)$$

For $\phi \in \mathrm{Aut}(\mathscr{A}_a)$ let $[\phi]$ denote its class in $\mathrm{Out}(\mathscr{A}_a)$. Then

$$\begin{aligned} H^A/H^{(V)} &= \mathrm{Out}^E(\mathscr{A}_a) \\ &= \left\{ x \in \mathrm{Out}(\mathscr{A}_a) \;\middle|\; \begin{array}{l} \exists \phi_a \in \mathrm{Aut}(\mathscr{A}_a)\ \phi_e \in \mathrm{Aut}(\mathscr{A}_e) \text{ such that,} \\ x = [\phi_a] \text{ and } \forall s \in \mathscr{A}_e,\ \phi_a(\alpha_e(s)) = \alpha_e(\phi_e(s)) \\ \text{and } \phi_a(\alpha_{\bar{e}}(s)) = \alpha_{\bar{e}}(\phi_e(s)) \end{array} \right\}, \end{aligned} \quad (6)$$

$$H^{(V)}/H^{(V,E)} = \frac{\mathrm{ad}(N_e) \cap \mathrm{ad}(N_{\bar{e}})}{\mathrm{ad}(\mathscr{A}_e)}, \quad (7)$$

$$H^{(V,E)}/H^{(V,E]} \cong \left(\frac{Z_e}{\alpha_e Z_{(e)} \cdot Z_a} \right) \times \left(\frac{Z_{\bar{e}}}{\alpha_{\bar{e}} Z_{(e)} \cdot Z_a} \right). \quad (8)$$

In Theorem 8.1(7), $Z_{aE} = Z_a \cap \alpha_e \mathscr{A}_e \cap \alpha_{\bar{e}} \mathscr{A}_e$, and the map $\omega : Z_{aE} \to (Z_{(e)}/Z_e(\mathfrak{A})) \times' (Z_{(\bar{e})}/Z_{\bar{e}}(\mathfrak{A}))$ is trivial. Since the second coordinate in the latter product is determined by the first, we see that

$$H^{(V,E]}/H^{(V,EZ]} \cong Z_{(e)}/Z_e(\mathfrak{A}) = Z(\mathscr{A}_e)/Z_e(\mathfrak{A}). \quad (9)$$

From (2) we can calculate

$$\begin{aligned} &Z_e(\mathfrak{A})(\cong Z(\Gamma)) \\ &\quad = \{s \in \mathscr{A}_e \mid \alpha_e(s) = \alpha_{\bar{e}}(s) \in Z(\mathscr{A}_a)\}. \end{aligned} \tag{10}$$

Finally, since $\pi_1(A, a) = \langle e \rangle \cong \mathbf{Z}$, it follows from Theorem 8.1(8) that

$$H^{(V,EZ]} \;\cong\; Z_e(\mathfrak{A}). \tag{11}$$

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